



# The parameterization method for invariant manifolds III: overview and applications

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## Abstract

We describe a method to establish existence and regularity of invariant manifolds and, at the same time to find simple maps which are conjugated to the dynamics on them. The method establishes several invariant manifold theorems. For instance, it reduces the proof of the usual stable manifold theorem near hyperbolic points to an application of the implicit function theorem in Banach spaces. We also present several other applications of the method.

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## 1. Introduction

The goal of this paper is to present a tutorial on “the parameterization method”, a technique recently introduced by the authors [CFdIL03a,CFdIL03b] to study invariant manifolds of dynamical systems. As a first simple application, the method allows to give quick proofs of stable and unstable manifold theorems. More importantly, it leads to new results on existence of invariant manifolds, as well as on their regularity and dependence on parameters.

To be more precise, the parameterization method allows to establish the existence of smooth invariant manifolds associated to linear subspaces invariant by the linearization and which satisfy some non-resonance conditions. As a novelty with respect to previous works, the invariant linear subspaces need not be spectral subspaces. Even further, they need not have an invariant complement.

The parameterization method lends itself to very efficient computer implementations since it provides a global representation of the manifold, and it also allows a very efficient discussion of dependence on parameters.

Some extensions of the method to quasi-periodic systems and numerical implementations are presented in [HdIL03b,HdIL03a]. In this paper we highlight the main geometric ideas and invariant objects obtained, as well as the main technical tools (Banach spaces with norms tailored to the problem, differentiability results of composition operators, implicit function and fixed point theorems, cohomology equations). Hence, we have not included the optimal regularity results, neither the technical ideas needed to obtain them. The interested reader may find these in [CFdIL03a,CFdIL03b].

Some variants of the method seem to have appeared in fragmentary form in the work of Poincaré on automorphic forms [Poi90], later in his research on dynamics, and also in the work on Lyapunov [Lya92]. Of course, modern techniques such as implicit function theorems on Banach spaces were not available at that time, which made these works quite fragmentary and full of restrictions. In some particular applications (specially in relation with numerical calculations), the method seems to have been rediscovered

several times, again under extra restrictions. In Appendix B we comment on these historical matters.

In Section 2 we describe the basic ideas and objects of the method, both for dynamical systems given by maps and for those given by ordinary differential equations. Section 3 describes the main result of [CFdlL03a], stated both for maps and for differential equations.

We have tried that each of the sections after Section 3 could be read independently of each other. Each of them presents a full proof of one result that illustrates some of the main ideas involved with the method. The applications in Part I (Sections 4–6) are simpler, while the results in Part II (Sections 7–10) are sharper and more delicate.

Finally, we have included two appendices with comments on cohomology equations, non-uniqueness of invariant manifolds (an important point when doing numerical computations), and historical remarks on the literature of the subject, including applications.

## 2. The parameterization method

### 2.1. The parameterization method for maps

Given a map  $F: U \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$  with  $F(0) = 0$ , where  $U$  is an open set containing the origin, a natural way to try to find a manifold invariant under  $F$  and modeled on a subspace  $E \subset \mathbb{R}^d$ , is to look for an embedding  $K: U_1 \subset E \rightarrow \mathbb{R}^d$  and a map  $R: U_1 \subset E \rightarrow U_1$  in such a way that

$$F \circ K = K \circ R. \quad (2.1)$$

The fact that the manifold  $K(U_1)$  passes through the origin is ensured by requiring

$$K(0) = 0. \quad (2.2)$$

The fact that the manifold is tangent at the origin to  $E$  is guaranteed by requiring

$$DK(0)E = E. \quad (2.3)$$

Note that (2.1) ensures that the range  $K(U_1)$  of  $K$  is invariant under  $F$ . We think of  $K$  as giving a parameterization of the invariant manifold  $K(U_1)$ . Moreover,  $R$  is the dynamics of  $F$  restricted to the invariant manifold.

Note also that differentiating (2.1) at the origin and using (2.2) and (2.3), we deduce

$$DF(0)E \subset E.$$

Thus,  $E$  must be an invariant subspace under the linearization  $DF(0)$  of  $F$  at the origin.

The fact that  $R$  is a representation (in some appropriate coordinates) of the dynamics of the map  $F$  restricted to the manifold, tells us that we need to consider it as part of the objects to be determined (or at least to be flexible about its choice) since, depending on the non-linear terms, the dynamics on the stable manifold (for instance on the classical stable manifold) may belong to different equivalence classes under smooth conjugacy.

An important observation is that if we consider

$$\mathcal{T}(F, K, R) := F \circ K - K \circ R, \quad (2.4)$$

and write Eq. (2.1) as  $\mathcal{T}(F, K, R) = 0$ , then  $\mathcal{T}$  is differentiable in  $K$  whenever  $K$  is given the topology of  $C^r$  spaces (provided that  $F$  is sufficiently differentiable). Hence, Eq. (2.1) can be studied via the standard implicit function theorem in Banach spaces, even in the standard  $C^r$  spaces. This leads very quickly and painlessly to some results on existence and differentiability with respect to parameters for finitely differentiable maps. This is the approach undertaken in this article (with the exception of Section 5, where we use the fixed point theorem for contractions). In [CFdlL03a] we used instead fixed point theory in jet spaces in order to obtain optimal regularity results.

The fact that  $\mathcal{T}$  is differentiable is in contrast with the functional equations that one has to deal with in the graph transform method, in which the operator whose fixed point gives the invariant graph is not differentiable in any of the classical  $C^r$  spaces, even if it is differentiable in spaces of analytic functions (see [Mey75]).

The linearized version of (2.4) with respect to  $K$  is formally (and rigorously, in regularity classes characterized below and in more generality in [dlLO99]):

$$D_2\mathcal{T}(F, K, R)\Delta = (DF \circ K)\Delta - \Delta \circ R. \quad (2.5)$$

The equations for  $\Delta$  obtained setting  $D_2\mathcal{T}(F, K, R)\Delta = \eta$  are called cohomology equations. We describe some aspects related to them in Appendix A. These equations have a very rich history (see e.g. [BdlW96]). Once a theory for the linearized equations is established, one can study the full non-linear equation (2.1) using a variety of methods (contraction mappings, implicit function theorems, deformation methods, etc.).

## 2.2. The parameterization method for flows

Very similar ideas to those used in the proof of the results for maps can be used to study invariant manifolds for differential equations

$$x' = \mathcal{X}(x).$$

Here  $\mathcal{X}$  is a vector field in  $U \subset \mathbb{R}^d$ , where  $U$  is an open set containing the origin, with  $\mathcal{X}(0) = 0$ . If  $E$  is a subspace of  $\mathbb{R}^d$  invariant by  $D\mathcal{X}(0)$ , we look for a parameterization

$K : U_1 \subset E \longrightarrow \mathbb{R}^d$  and a vector field  $R$  in  $U_1 \subset E$  such that

$$\mathcal{X} \circ K = DK \cdot R. \quad (2.6)$$

That is, we ask the vector field  $\mathcal{X}$  on the image of  $K$  to be the pull forward of the vector field  $R$  in  $E$ . Eq. (2.6) expresses that at the range of  $K$ , the vector field  $\mathcal{X}$  is tangent to the range of  $K$ . Hence, the range of  $K$  is invariant under the flow of  $\mathcal{X}$ . Moreover, the vector field  $R$  is the representation in parameters of the restriction of  $\mathcal{X}$  to the invariant manifold. This direct study, which will be described in Section 10, is illuminating and, for practical calculations advantageous. Nevertheless those interested mainly in existence and regularity results may prefer an abstract argument which shows that the results for differential equations follow from the results for maps. This indirect approach is as follows:

If  $\{\varphi_t\}_{t \in \mathbb{R}}$  is the flow associated to  $\mathcal{X}$  and  $W$  is a manifold such that  $\varphi_1(W) \subset W$  (i.e.,  $W$  is an invariant manifold for the map  $\varphi_1$ ), then we have

$$\varphi_1(\varphi_t(W)) = \varphi_t(\varphi_1(W)) \subset \varphi_t(W). \quad (2.7)$$

If the invariant manifold theorem for the map  $\varphi_1$  includes local uniqueness under hypothesis that are invariant under the evolution by  $\varphi_t$ , since from (2.7) we obtain that  $\varphi_t(W)$  is also invariant under  $\varphi_1$ , we conclude that

$$\varphi_t(W) \subset W.$$

That is,  $W$  is also invariant under  $\varphi_t$ , for all  $t > 0$ .

We point out that in all discussions in this section it makes no difference to replace  $\mathbb{R}^d$  by a Banach space  $X$ . In the following sections we work in  $\mathbb{R}^d$  except in Sections 6 and 8, where we deal with invariant manifolds in Banach spaces.

### 3. Main results

In this section we present the statement of two theorems on non-resonant invariant manifolds associated to a fixed point. We have selected them as representatives of the results of the paper. However, in the paper we deal with other results which range from simpler to more difficult situations, and also with an invariant manifold theorem associated to periodic orbits of vector fields.

To state the results, we first recall some standard terminology. The spectrum of a linear operator  $A$  in  $\mathbb{R}^d$  will be denoted by  $\text{Spec}(A)$ . We emphasize that  $\text{Spec}(A)$  denotes the spectrum of the complex extension of  $A$ , and hence  $\text{Spec}(A)$  is a compact subset of  $\mathbb{C}$ . For  $j \in \mathbb{N}$  and  $S \subset \mathbb{C}$ , we use the notation

$$jS := \{a_1 + \cdots + a_j \mid a_i \in S\}.$$

We use a similar notation for sum, difference, and product of sets.

We say that a function is in  $C^\omega$  if it is analytic.

### 3.1. Manifolds associated to a fixed point of a map

The heuristic idea is that, given a map  $F$  in  $\mathbb{R}^d$ , with  $F(0) = 0$ , to every linear subspace  $E \subset \mathbb{R}^d$  invariant under  $DF(0)$ , there should correspond a smooth manifold invariant under the map  $F$ , passing through the origin and tangent there to  $E$ . Of course, this should be true if the map  $F$  is smoothly linearizable but, as the stable (or strong stable) manifold theorems show, the hypothesis of linearizability is much stronger than needed. Nevertheless, as shown in examples in [dIL97], some non-resonance conditions are necessary for the existence of a smooth invariant manifold.

The non-resonance conditions (hypothesis 3) in the following theorem consist of certain hypotheses on the spectrum of  $DF(0)$ . They are automatically satisfied when dealing with the stable or strong stable manifold theorems.

The following is the result concerning non-resonant invariant manifolds for maps. Its proof is given in Section 9.

**Theorem 3.1.** *Let  $F: U \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a  $C^{r+1}$  map in a neighborhood  $U$  of the origin, with  $F(0) = 0$ , and  $r \in \mathbb{N} \cup \{\omega\}$ .*

*Denote  $A = DF(0)$ . Let  $L \in \mathbb{N}$ ,  $L \geq 1$ . Assume that:*

- (1) *There is a linear subspace  $E$  of  $\mathbb{R}^d$  such that  $A(E) \subset E$ . Hence there is a decomposition  $\mathbb{R}^d = E \oplus C$  and, with respect to it,  $A$  has the form*

$$A = \begin{pmatrix} A_E & B \\ 0 & A_C \end{pmatrix}. \quad (3.1)$$

- (2)  $\|A_E\| < 1$ .  
 (3)  $\text{Spec}(A_E)^j \cap \text{Spec}(A_C) = \emptyset$  for  $j = 2, \dots, L$ .  
 (4)  $A$  is invertible.  
 (5)  $(\text{Spec}(A_E))^{L+1} \text{Spec}(A^{-1}) \subset \{z \in \mathbb{C} \mid |z| < 1\}$ .  
 (6)  $L + 1 \leq r$ .

*Then, there exist a  $C^r$  map  $K: U_1 \subset E \rightarrow \mathbb{R}^d$ , where  $U_1$  is an open neighborhood of 0 in  $E$ , and a polynomial  $R: E \rightarrow E$  of degree at most  $L$ , such that*

$$\begin{aligned} F \circ K &= K \circ R \quad \text{in } U_1, \\ K(0) &= 0, \quad DK(0)E = E, \\ R(0) &= 0, \quad DR(0) = A_E. \end{aligned} \quad (3.2)$$

Theorem 3.1 is a simplified, but close to optimal, version of the main result of [CFdIL03a]. In that paper, the result was stated in general Banach spaces instead of just in  $\mathbb{R}^d$ . The regularity results on dependence with respect to parameters were studied in [CFdIL03b].

We remark that the loss of one derivative in the regularity of the manifold as stated in Theorem 3.1 (in which  $F$  is assumed to be  $C^{r+1}$ , but the manifold obtained is just  $C^r$ ), can be improved. In [CFdIL03a] the manifold is proved to have the same finite differentiability as the map, that is, no derivative is lost. To obtain this sharp result one needs to consider Eq. (2.1) as a fixed point problem enjoying special properties, and study in detail the convergence to the limit—rather than applying the implicit function theorem as we do in the present paper.

### 3.2. Manifolds associated to a fixed point of a vector field

As in the case for maps, if  $\mathcal{X}$  is a vector field in  $U \subset \mathbb{R}^d$ , with  $\mathcal{X}(0) = 0$ , and  $E \subset \mathbb{R}^d$  is a linear subspace invariant under  $D\mathcal{X}(0)$ , there may correspond a smooth manifold invariant under the flow of  $\mathcal{X}$ , passing through the origin and tangent to  $E$  at it.

The analogous remarks of the previous subsection apply in this case.

The corresponding result is the following theorem. Its proof is given in Section 10.

**Theorem 3.2.** *Let  $\mathcal{X}$  be a  $C^{r+1}$  vector field on an open set  $U$  of  $\mathbb{R}^d$  with  $0 \in U$ , such that  $\mathcal{X}(0) = 0$  and  $r \in \mathbb{N} \cup \{\omega\}$ . Let  $A = D\mathcal{X}(0)$  and  $L \in \mathbb{N}$ ,  $L \geq 1$ . Suppose that:*

- (1) *There is a linear subspace  $E$  of  $\mathbb{R}^d$  such that  $A(E) \subset E$ . Hence there is a decomposition  $\mathbb{R}^d = E \oplus C$  and, with respect to it,  $A$  has the form*

$$A = \begin{pmatrix} A_E & B \\ 0 & A_C \end{pmatrix}.$$

- (2)  $\text{Spec}(A_E) \subset \{z \in \mathbb{C} \mid \text{Re } z < 0\}$ .  
 (3)  $j \text{Spec}(A_E) \cap \text{Spec}(A_C) = \emptyset$  for  $j = 2, \dots, L$ .  
 (4)  $\text{Spec}(-A) + (L+1)\text{Spec}(A_E) = \{-\lambda + \mu_1 + \dots + \mu_{L+1} \mid \lambda \in \text{Spec}(A) \text{ and } \mu_1, \dots, \mu_{L+1} \in \text{Spec}(A_E)\} \subset \{z \in \mathbb{C} \mid \text{Re } z < 0\}$ .  
 (5)  $L+1 \leq r$ .

*Then, there exist a  $C^r$  map  $K: U_1 \subset E \rightarrow \mathbb{R}^d$ , where  $U_1$  is a neighborhood of 0 in  $E$ , and a polynomial  $R: E \rightarrow E$  of degree at most  $L$ , such that*

$$\mathcal{X} \circ K = DK \cdot R \quad \text{in } U_1, \quad (3.3)$$

$$K(0) = 0, \quad DK(0)E = E, \quad (3.4)$$

$$R(0) = 0, \quad DR(0) = A_E. \quad (3.5)$$

Note that (3.3) ensures that the image  $K(U_1)$  of  $K$  is invariant by the flow of  $\mathcal{X}$ . Condition (3.4) ensures that  $K(U_1)$  passes through the origin and it is tangent to  $E$  there.

We emphasize that the subspace  $E$  need not have an invariant complement. In addition, the spectrum of  $A_E$  and that of  $A_C$  need not be disjoint, since condition (3) is only required for indices  $j$  bigger or equal than 2. For example, the theorem applies to

$$A = \begin{pmatrix} -2 & 1 & & & \\ 0 & -2 & & & \\ & & -3 & & \\ & & & -5 & 1 \\ & & & 0 & -5 \end{pmatrix}.$$

Then, denoting by  $E_i$  the  $i$ th coordinate axis, we could associate invariant manifolds to  $E_1$ ,  $E_3$ ,  $E_4$ ,  $E_1 \oplus E_2$ ,  $E_4 \oplus E_5$ , or to sums of these spaces, e.g.,  $E_1 \oplus E_4$ ,  $E_1 \oplus E_2 \oplus E_4$ , etc.

On the other hand, the result does not apply to  $E_1 \oplus E_3$  since the eigenvalues corresponding to  $E_1$  and  $E_3$  when added give an eigenvalue in the complement. Indeed, if we consider the non-linear part  $x_1 x_3 e_5$ , where  $e_5$  is the fifth element of the canonical basis, it is easy to see that there is no  $C^2$  invariant manifold tangent to  $E_1 \oplus E_3$ .

**Remark 3.3.** Note that the manifold associated to the eigenvalue  $-3$  is not normally hyperbolic in the sense of [Fen72,HPS77] since the exponents along the manifold do not dominate the others. Also, if we consider the manifold associated to the eigenvalues  $-2$  and  $-5$ , the complement contains the exponent  $-3$  which is between the tangential exponents. The existence of this manifold does not follow from normal hyperbolicity and the proof depends on the fact that it is attached to a fixed point.

## Part I

### 4. Analytic one-dimensional stable manifolds

In this section we prove a theorem that serves as motivation for several results later. Indeed, the main ideas of future results appear here.

As a matter of fact, a very similar theorem had been proved by Poincaré [Poi90], which used a very different method (the majorant method) from the one used here. Clearly, the methods used here (Banach spaces and implicit function theorem) were not available at the time of [Poi90]. The theorem has been rediscovered in different guises in the literature, specially in relation with numerical calculations (see Appendix B).

**Theorem 4.1.** Let  $F : U \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an analytic map in a neighborhood  $U$  of 0, with  $F(0) = 0$ . Let  $\lambda \in \mathbb{R}$  be an eigenvalue of  $A := DF(0)$ , and let  $v \in \mathbb{R}^d \setminus \{0\}$  satisfy  $Av = \lambda v$ . Assume:

- (1)  $A$  is invertible.
- (2)  $0 < |\lambda| < 1$ .
- (3)  $\lambda^n \notin \text{Spec}(A)$  for every integer  $n \geq 2$ .



Then, there exists an analytic map  $K : U_1 \subset \mathbb{R} \rightarrow \mathbb{R}^d$ , where  $U_1$  is an open neighborhood of 0 in  $\mathbb{R}$ , satisfying

$$F(K(x)) = K(\lambda x) \quad \text{in } U_1, \quad (4.1)$$

$K(0) = 0$ , and  $K'(0) = v$ . Therefore, the image of  $K$  is an analytic one-dimensional manifold invariant under  $F$  and tangent to  $v$  at the origin. Moreover, the dynamics on the invariant manifold is conjugated to the linear map  $x \mapsto \lambda x$  in the space of parameters.

In addition, if  $\widehat{K}$  is another analytic solution of  $F \circ K = K \circ \lambda$  in a neighborhood of the origin, with  $\widehat{K}(0) = 0$  and  $\widehat{K}'(0) = \beta K'(0)$  for some  $\beta \in \mathbb{R}$ , then  $\widehat{K}(t) = K(\beta t)$  for  $t$  small enough.

**Remark 4.2.** In the statement of the theorem,  $\text{Spec}(A)$  denotes the spectrum of  $A$ . Note that, since  $0 \notin \text{Spec}(A)$  by (1), we have that  $\text{Spec}(A)$  excludes a ball of radius  $\rho$ . By (2), there is an integer  $n_0$  such that  $|\lambda|^{n_0} < \rho$ . Now, if  $n \geq n_0$  then condition (3) holds.

Hence, hypothesis (3), even if it seems to require infinitely many conditions, all but  $n_0$  of them are always fulfilled. Note also that  $n_0$  is constant in open neighborhoods of  $A$ . This shows that hypothesis (3) (which is called a non-resonance condition) fails only on a manifold of maps  $F$  of finite codimension.

It turns out that condition (3) is necessary for having a manifold as the one claimed in the statement. Indeed, consider the map  $(x, y) \mapsto (\frac{1}{2}x, \frac{1}{4}y + x^2)$ . An invariant manifold tangent to the vector  $(1, 0)$  can be put as a graph of a function  $\varphi : U_1 \subset \mathbb{R} \rightarrow \mathbb{R}$  satisfying the invariance condition  $\varphi(x/2) = \varphi(x)/4 + x^2$ . Such  $\varphi$  cannot be  $C^2$  because taking two derivatives on both sides of the previous condition we get a contradiction.

**Remark 4.3.** By considering  $F^{-1}$  and  $\lambda^{-1}$  in place of  $F$  and  $\lambda$ , hypothesis (2) in Theorem 4.1 can be changed to  $|\lambda| > 1$ .

**Remark 4.4.** Note that we do not require  $\lambda$  to be a simple eigenvalue. We could have that  $\lambda$  has other eigenvectors, linearly independent of  $v$ , or that  $v$  is an eigenvector in a non-trivial Jordan block. In the latter case, note that the invariant eigenspace generated by  $v$  does not have an invariant complement.

**Remark 4.5.** Note that if  $F^{-1}$  is entire (for instance, if  $F^{-1}$  is a polynomial as it happens for the Hénon map), then  $K$  is an entire function. Indeed, when  $F^{-1}$  is entire, if  $K$  is defined on a ball  $B_\rho$ , (4.1) shows that  $K = F^{-1} \circ K(\lambda \cdot)$  is defined on  $\lambda^{-1}B_\rho = B_{\lambda^{-1}\rho}$ . Repeating the argument, the domain of definition of  $K$  becomes the whole plane.

This was the motivation in [Poi90], namely, to construct entire functions which satisfied polynomial “duplication of angle formulas” similar to the familiar formulas for  $\sin$ ,  $\cos$ , or for elliptic functions. The argument we present here leads to the construction

of functions whose “double angle” values can be expressed—through (4.1)—as a given function of those of the “single angle”.

One particularly interesting case of the situations covered by Theorem 4.1 is when  $\|A\| < 1$ ,  $\lambda$  is simple and it is the eigenvalue of  $A$  closest to the unit circle. Note that, upon iteration of  $A$ , the component along  $v$  is the one that decays more slowly and hence, the one which controls the asymptotic behavior. The invariant manifold associated to this eigenvalue is a non-linear analogue and can also be used to study the asymptotic behavior of the iterates of  $F$ . It is usually called a slow manifold.

**Proof of Theorem 4.1.** Using power series,  $F$  can be considered as an analytic function in a neighborhood of 0 in  $\mathbb{C}^d$ . Following the main idea of the parameterization method, we try to find  $K : D \subset \mathbb{C} \rightarrow \mathbb{C}^d$  such that

$$F \circ K(z) - K(\lambda z) = 0 \quad (4.2)$$

for  $z \in D$ , where  $D$  is the unit disk of  $\mathbb{C}$ . We write  $F(\zeta) = A\zeta + N(\zeta)$  with  $A = DF(0)$ .

If we try to solve (4.2) equating powers of  $z$  on both sides, we obtain that  $K(z) = \sum_{n \geq 1} K_n z^n$  should satisfy

$$\begin{aligned} AK_1 &= \lambda K_1, \\ AK_n + R_n(K_1, \dots, K_{n-1}) &= \lambda^n K_n, \quad n \geq 2, \end{aligned} \quad (4.3)$$

where  $R_n$  is a polynomial expression obtained expanding the composition in (4.2).

The first equation in (4.3) does not determine  $K_1$  completely, only tells us that  $K_1$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ . We take  $K_1$  to be a multiple of  $v$  such that  $|K_1| = \delta$ , where  $\delta$  is small enough (we will give the precise smallness conditions on  $\delta$  later, as they appear in the proof).

Once we have chosen  $K_1$ , (4.3) allows to determine in a unique fashion all the other  $K_n$ 's, as

$$K_n = -(A - \lambda^n)^{-1} R_n(K_1, \dots, K_{n-1}), \quad n \geq 2. \quad (4.4)$$

The inverse used in (4.4) exists by assumption (3). One can indeed show, studying directly the recursion in (4.4), that the  $K_n$  thus defined lead to an analytic function. This was the approach used in [Poi90].

We will, however, follow another route to study (4.2). We use techniques of functional analysis which can be adapted to other settings, such as dealing with a finitely differentiable  $F$ , or treating maps  $F$  defined in Banach spaces.

We write  $K(z) = K_1 z + K^>(z)$ , and recall that we have already picked  $K_1$  and that it is small. The equation for  $K^>(z)$  reads

$$AK^>(z) + N(K_1 z + K^>(z)) - K^>(\lambda z) = 0. \quad (4.5)$$

We consider  $K^>$  belonging to the Banach space  $H$  of analytic functions in the unit disk, vanishing at the origin along with their first derivative, and with the following norm being finite:

$$H = \left\{ K^> : D \subset \mathbb{C} \rightarrow \mathbb{C}^d \left| K^>(z) = \sum_{n=2}^{\infty} K_n z^n, \|K^>\| := \sum_{n=2}^{\infty} |K_n| < \infty \right. \right\}.$$

We recall that the analytic functions  $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  such that  $\|f\| := \sum_{n \geq 0} |f_n| < \infty$  form a Banach algebra with the previous norm (see, e.g., [Car95] for a straightforward proof). Indeed, it suffices to apply the triangle inequality in the expression for the coefficients of the product  $(fg)_n = \sum_{i+j=n} f_i g_j$  and then sum in  $n$ . Of course, the ideal  $H = \{f \mid f_0 = f_1 = 0\}$  is also a Banach algebra.

We can reformulate (4.5) as an operator equation

$$\mathcal{T}(K_1, K^>) = 0, \quad (4.6)$$

where  $\mathcal{T}$  is the non-linear operator  $\mathcal{T} : V \subset \mathbb{C}^n \times H \rightarrow H$  defined by

$$\mathcal{T}(K_1, K^>)(z) := (SK^>)(z) + N(K_1 z + K^>(z)), \quad (4.7)$$

where  $(S\Delta)(z) = A\Delta(z) - \Delta(\lambda z)$  and  $V$  is a small neighborhood of  $(0, 0)$  in  $\mathbb{C}^n \times H$  to be determined later.

The main properties of  $\mathcal{T}$  for our purposes are summarized in the following:

**Proposition 4.6.** *If  $V$  is contained in a ball of  $\mathbb{C}^n \times H$  centered at  $(0, 0)$  and of sufficiently small radius, then:*

- (1) *The operator  $\mathcal{T} : V \subset \mathbb{C}^n \times H \rightarrow H$  is well defined and analytic.*
- (2)  *$D_2 \mathcal{T}(0, 0) = S$ .*

**Proof.** Note that  $\sup_{|z| < 1} |K(z)| \leq |K_1| + \|K^>\|$ . Hence, if we take  $|K_1| + \|K^>\|$  smaller than the radius of analyticity of  $N$ , then  $N(K_1 z + K^>(z))$  is analytic in the unit disk  $D$ . Note also that  $N(K_1 z + K^>(z)) = O(z^2)$  because  $N$  vanishes with its derivative at the origin.

Since  $N$  is analytic, denoting by  $N^i$  its  $i$ th component, we may use multi-index notation to write:

$$N^i(\zeta) = \sum_{k \in \mathbb{Z}_+^d} N_k^i \zeta^k,$$

where

$$|N_k^i| \leq \rho^{|k|}$$

for some  $\rho > 0$ . Therefore, the series

$$\sum_{k \in \mathbb{Z}_+^d} N_k^i (K_1 z + K^>(z))^k$$

converges in the Banach algebra norm of  $H$ , and hence  $\mathcal{T}$  is analytic as an operator, provided that  $|K_1| + \|K^>\| \leq \rho/2$ . This is a condition on the size of the neighborhood  $V$ .

The second statement of the proposition is elementary.  $\square$

**Lemma 4.7.** *The operator  $S$  acting on  $H$  and defined by*

$$(S\Delta)(z) = A\Delta(z) - \Delta(\lambda z),$$

*is boundedly invertible in  $H$ .*

**Proof.** Given  $\eta \in H$  with  $\eta(z) = \sum_{n=2}^{\infty} \eta_n z^n$ , we look for  $\Delta \in H$  with  $\Delta(z) = \sum_{n=2}^{\infty} \Delta_n z^n$ , such that

$$S\Delta = \eta.$$

Equating coefficients, we are lead to

$$A\Delta_n - \Delta_n \lambda^n = \eta_n, \quad n \geq 2.$$

Hence  $\Delta_n = (A - \lambda^n)^{-1} \eta_n$ . By hypotheses (1)–(3), we have that  $\|(A - \lambda^n)^{-1}\| \leq C$  for some constant  $C$  independent of  $n$ . We conclude that  $\|\Delta\| \leq C\|\eta\|$ .  $\square$

Theorem 4.1 follows immediately from Proposition 4.6 and Lemma 4.7, applying the standard implicit function theorem in Banach spaces—see [Nir01]—to (4.6). We conclude that for every small enough multiple  $K_1$  of  $v$ , we can obtain the corresponding  $K^>$ . Then, the image of  $K = K_1 + K^>$  gives the invariant manifold claimed in the statement.

If  $K(z)$  satisfies (4.2) and  $\sigma \in \mathbb{C}$ , then  $\widehat{K}(z) = K(\sigma z)$  also satisfies (4.2). Note that  $\widehat{K}_1 = \sigma K_1$ . This explains the lack of uniqueness in  $K_1$ . That is, if we choose  $K_1$  differing only by a multiple, by the uniqueness given by the implicit function theorem we are only choosing another parameterization of the same manifold, related to it by a linear change of scale. This is the last statement of the Theorem 4.1.  $\square$

**Remark 4.8.** Using the implicit function theorem we see immediately that we have analytic dependence of  $K^>$  on  $A$ ,  $N$ , and  $K_1$ , considered as independent parameters. Of course,  $A$  and  $K_1$  are always related by the fact that  $K_1$  is an eigenvector of  $A$ .

If we choose  $K_1$  to depend analytically on  $A$  (for instance, if  $\lambda$  is a simple eigenvalue, we can choose  $K_1$  by requesting  $|K_1| = \delta$  and to be continuous with respect to  $A$ ), then we obtain analytic dependence of  $K$  with respect to  $A$  and  $N$ .

**Remark 4.9.** We recall that the proof of the implicit function theorem consists essentially in using the existence of  $S^{-1}$  to transform (4.6) into

$$K^>(z) = -S^{-1}N(K_1 z + K^>(z)).$$

The right-hand side of this fixed point equation is a contraction (as a function of  $K^>$ ). This gives a practical algorithm.

## 5. One-dimensional stable directions around periodic orbits of analytic differential equations

In this section we discuss a situation that appears rather frequently in applications and where periodic orbits play an important role. We consider an analytic vector field  $\mathcal{X}$  in  $\mathbb{R}^d$ , assume that we are given a periodic orbit  $\gamma(t)$ , and try to find two-dimensional invariant manifolds containing  $\gamma$ .

From a strictly mathematical point of view, the result of this section (Theorem 5.4) is equivalent to the results of Section 4. By considering a return map to a manifold transversal to the flow, it is easy to see that results for maps imply results for differential equations. Conversely, using suspensions, results for differential equations imply results for maps. Nevertheless, we warn that for problems that appear in practice, there are differences in the analytic properties of the solutions (see Remark 5.11).

On the other hand, since differential equations appear often in practice and the language of differential equations is more familiar to many practitioners, we believe interesting to present a detailed proof for ordinary differential equations.

Moreover the proof presented in this section leads immediately to practical numerical algorithms. See [Cap04] for an implementation in a problem of celestial mechanics.

### 5.1. Formal solution and heuristic discussion

We start by discussing the problem of finding a solution in terms of formal power series, ignoring for the moment issues of convergence. This will set some of the notation to be used, and also serve as a motivation for some of the hypotheses to be made later in the precise formulation of the results.

We consider an analytic vector field  $\mathcal{X}$  on  $\mathbb{R}^d$  and assume that we are given a map  $\gamma: \mathbb{T}^1 \rightarrow \mathbb{R}^d$  (the periodic orbit, parameterized to have period 1) and a number  $T \in \mathbb{R}^+$  (the period of the orbit) such that

$$\frac{1}{T} \frac{d}{d\theta} \gamma(\theta) = \mathcal{X}(\gamma(\theta)). \quad (5.1)$$

We seek a map  $K : \mathbb{T}^1 \times U \subset \mathbb{T}^1 \times \mathbb{R} \rightarrow \mathbb{R}^d$ , where  $U$  is an open interval containing 0, and a number  $\lambda$  in such a way that

$$\left( \frac{1}{T} \frac{\partial}{\partial \theta} + \frac{\lambda \sigma}{T} \frac{\partial}{\partial \sigma} \right) K(\theta, \sigma) = \mathcal{X}(K(\theta, \sigma)). \quad (5.2)$$

We also ask that  $K(\theta, 0) = \gamma(\theta)$  for all  $\theta$ .

Clearly, Eq. (5.2) says that  $\mathcal{X}$  evaluated at points in  $\mathcal{M} := \text{Range}(K)$  is a vector tangent to  $\mathcal{M}$ . Hence,  $\mathcal{M}$  is an invariant manifold under the flow of  $\mathcal{X}$ . Moreover, in the variables  $(\theta, \sigma)$  parameterizing  $\mathcal{M}$ , the motion generated by  $\mathcal{X}$  on  $\mathcal{M}$  is given by

$$\begin{aligned} \dot{\theta} &= 1/T, \\ \dot{\sigma} &= \lambda \sigma / T. \end{aligned} \quad (5.3)$$

That is, the variable  $\theta$  keeps on rotating at speed  $1/T$ , while the variable  $\sigma$  moves in an exponential way. Note that the vector field in (5.3) is the vector field that we named  $R$  in (2.6). With this choice of  $R$ , Eq. (2.6) in our general presentation becomes the current (5.2).

The orbit of the point  $K(\theta_0, \sigma_0)$  is  $K(\theta_0 + t/T, \sigma_0 e^{\lambda t/T})$ . Hence, if  $\lambda < 0$ , the orbit of  $K(\theta_0, \sigma_0)$  approaches exponentially fast the orbit of  $K(\theta_0, 0)$ , which is a point in the original periodic orbit. We therefore see that  $\{K(\theta_0, \sigma) \mid \sigma \in U\}$  is contained in the stable manifold  $W_{\gamma(\theta_0)}^s$  associated to the point  $\gamma(\theta_0)$  in the sense that the difference of the orbit of a point in  $W_{\gamma(\theta_0)}^s$  and the orbit of  $\gamma(\theta_0)$  goes to 0. Moreover  $W_{\gamma}^s = \bigcup_{\theta_0 \in [0,1]} W_{\gamma(\theta_0)}^s$ .

**Remark 5.1.** Note that since  $R(U_1) \subset U_1$ , Eq. (5.2) does not have a unique solution. Indeed, if  $K$  is a solution then  $\tilde{K}(\theta, \sigma) = K(\theta + a, \sigma b)$  is also a solution, for every  $a$  and  $b$ . The meaning of  $a$  is the choice of origin of time, and  $b$  corresponds to the choice of units in  $\sigma$ . These ambiguities can be used to make  $K$  satisfy some normalization conditions.

In order to compute  $K$ , at least formally, we seek  $K$  as a power series

$$K(\theta, \sigma) = \sum_{n=0}^{\infty} K_n(\theta) \sigma^n. \quad (5.4)$$

Substituting (5.4) into (5.2), we match the terms with the same powers of  $\sigma$  and solve the resulting equations to obtain a formal solution of (5.2). We postpone the considerations of convergence to Section 5.2.

Equating the coefficients of  $\sigma^0$  on both sides of (5.2) we obtain

$$\frac{1}{T} \frac{d}{d\theta} K_0(\theta) = \mathcal{X} \circ K_0(\theta). \quad (5.5)$$

Eq. (5.5) admits the solution

$$K_0(\theta) = \gamma(\theta).$$

This is consistent with  $K(\theta, 0) = \gamma(\theta)$ , that is,  $\mathcal{M}$  contains indeed the periodic orbit. Of course, as observed in Remark 5.1, we could take solutions  $K_0(\theta) = \gamma(\theta + a)$ .

Equating coefficients of  $\sigma^1$  in (5.2) we obtain

$$\frac{1}{T} \frac{d}{d\theta} K_1(\theta) + \frac{\lambda}{T} K_1(\theta) = D\mathcal{X} \circ K_0(\theta) K_1(\theta). \quad (5.6)$$

Since equations of very similar form to (5.6) will often appear, we introduce the operator  $L$  defined by

$$LK_1(\theta) := \frac{d}{d\theta} K_1(\theta) - TD\mathcal{X} \circ K_0(\theta) K_1(\theta). \quad (5.7)$$

Eq. (5.6) amounts to  $K_1$  being an eigenfunction of  $L$  with eigenvalue  $-\lambda$ . Therefore, we conclude that to guarantee (5.6), we need to take  $-\lambda$  in the set of eigenvalues of  $L$ , and then  $K_1$  satisfying

$$LK_1 = -\lambda K_1.$$

We note that  $L$  is the operator which appears when solving the linearized (or variational) equations. In the next subsection we review the rather well known theory of solutions of these equations.

#### 5.1.1. Solutions of the linearized equations

The goal is to study the solvability of equations related to the linearized operator  $L$  and to obtain estimates on the solutions.

We denote by  $\Phi_\theta$  the fundamental solution of

$$L\Delta := \left( \frac{d}{d\theta} - TD\mathcal{X} \circ K_0 \right) \Delta = \left( \frac{d}{d\theta} - TD\mathcal{X} \circ \gamma \right) \Delta = 0. \quad (5.8)$$

That is,  $\Delta$  is a solution of (5.8) with  $\Delta(0) = \alpha$  if and only if  $\Delta(\theta) = \Phi_\theta \alpha$ . The matrix  $\Phi_\theta$  satisfies  $L\Phi = 0$ ,  $\Phi_0 = \text{Id}$ .

The matrix  $\Phi_1$  is called the monodromy matrix. Note that if we are going to find a solution  $\Delta$  of the linearized equation as a geometric object along the periodic orbit, it must satisfy  $\Delta(\theta + 1) = \Delta(\theta)$ . Hence, the monodromy matrix appears very often as an obstruction for the existence of geometric objects.

We note that  $\Phi_\theta e^{-\mu\theta}$  satisfies

$$\frac{d}{d\theta}(\Phi_\theta e^{-\mu\theta}) = T D\mathcal{X} \circ K_0(\theta) \Phi_\theta e^{-\mu\theta} - \mu \Phi_\theta e^{-\mu\theta}.$$

Hence  $\Phi_\theta e^{-\mu\theta}$  is the fundamental solution of  $(L + \mu)\Delta = 0$ . Thus, there is a non-trivial periodic solution  $\Delta$  of

$$(L + \mu)\Delta = 0 \quad (5.9)$$

if and only if we can find  $\alpha \neq 0$  such that

$$\alpha = \Delta(0) = \Delta(1) = \Phi_1 e^{-\mu} \Delta(0).$$

That is, we can find a non-trivial periodic solution of the homogeneous equation (5.9) if and only if  $e^\mu$  is an eigenvalue of the monodromy matrix. Conversely, if  $e^\mu$  is not an eigenvalue of the monodromy matrix, then  $L$  has zero null space among periodic functions, and thus we have the following existence and uniqueness result:

**Proposition 5.2.** *Assume that  $e^\mu$  is not an eigenvalue of the monodromy matrix  $\Phi_1$ . Then, given any periodic function  $R$ , there exists a unique periodic function  $\Delta$  solving the equation*

$$(L + \mu)\Delta = R. \quad (5.10)$$

An expression for the solution is given by

$$\Delta(\theta) = -\Phi_\theta \int_\theta^1 \Phi_s^{-1} e^{\mu(s-\theta)} R(s) ds + \Phi_\theta \Phi_1^{-1} e^{\mu(1-\theta)} \Delta(0), \quad (5.11)$$

where  $\Delta(0)$  solves

$$(\Phi_1 - e^\mu \text{Id}) \Phi_1^{-1} \Delta(0) = - \int_0^1 \Phi_s^{-1} e^{\mu s} R(s) ds. \quad (5.12)$$

**Proof.** Denote by  $\tilde{\Phi}_\theta = \Phi_\theta e^{-\mu\theta}$ , which is the fundamental solution of the homogeneous equation  $(L + \mu)\tilde{\Phi} = 0$ . The variation of parameters formula tells that all the solutions of (5.10) are

$$\begin{aligned} \Delta(\theta) &= \tilde{\Phi}_\theta \int_0^\theta \tilde{\Phi}_s^{-1} R(s) ds + \tilde{\Phi}_\theta \Delta(0) \\ &= \Phi_\theta e^{-\mu\theta} \int_0^\theta \Phi_s^{-1} e^{\mu s} R(s) ds + \Phi_\theta e^{-\mu\theta} \Delta(0). \end{aligned} \quad (5.13)$$



Since we require  $\Delta(1) = \Delta(0)$ , we are led to

$$(\text{Id} - \Phi_1 e^{-\mu})\Delta(0) = \Phi_1 e^{-\mu} \int_0^1 \Phi_s^{-1} e^{\mu s} R(s) ds. \quad (5.14)$$

This formula leads directly to (5.12) in the statement of the proposition.

Next, we rewrite (5.13) as

$$\Delta(\theta) = \Phi_\theta e^{-\mu\theta} \left\{ \int_0^1 \Phi_s^{-1} e^{\mu s} R(s) ds - \int_\theta^1 \Phi_s^{-1} e^{\mu s} R(s) ds \right\} + \Phi_\theta e^{-\mu\theta} \Delta(0). \quad (5.15)$$

Using (5.14), we can rewrite the first integral in (5.15) as  $\Phi_1^{-1} e^{\mu} (\text{Id} - \Phi_1 e^{-\mu})\Delta(0)$ . Inserting this expression in (5.15), we finally obtain formula (5.11) in the statement of the proposition.  $\square$

We have seen that  $K_1(\theta)$  is a periodic eigenfunction of  $L$  with eigenvalue  $-\lambda$  if and only if  $K_1(0)$  is an eigenvector of the monodromy matrix  $\Phi_1$  with eigenvalue  $e^\lambda$ . Once  $K_1(0)$  is chosen as an eigenvector,  $K_1(\theta)$  is determined for all  $\theta \in \mathbb{T}^1$  by solving Eq. (5.6), that is,  $K_1(\theta) = e^{\mu(1-\theta)} \Phi_\theta \Phi_1^{-1} K_1(0)$ .

It is important to note that the solution of Eq. (5.6) is therefore a finite-dimensional problem (through the consideration of the monodromy matrix), in spite of the fact that it naively looks like an infinite-dimensional problem involving the unbounded differential operator  $d/d\theta$ .

**Remark 5.3.** There is a free multiplicative factor in the choice of  $K_1(0)$ , and hence of  $K_1(\theta)$ . The choice of this multiple corresponds to the choice of the parameter  $b$  in Remark 5.1, and hence it can be done in any way we like. In the proofs of convergence, it will be convenient to choose the multiple small enough, so that we can assume that  $\sigma$  ranges in the unit ball of the complex plane.

In the numerical implementations,  $b$  can be chosen to reduce the roundoff error. It is convenient to choose it so that all the coefficients  $K_n$  are of size 1. In practice, one can do a preliminary run which allows to get a reasonable estimate of  $b$ . Then, a fuller run with the appropriate  $b$  is numerically more reliable.

### 5.1.2. Formal solution

Once we have chosen the terms of order zero and order one, we study the equations obtained by matching terms of  $\sigma^n$ ,  $n \geq 2$ , in (5.2). We obtain

$$\frac{1}{T} \frac{d}{d\theta} K_n + \frac{n\lambda}{T} K_n = D\mathcal{X} \circ K_0 K_n + R_n,$$

where  $R_n$  is a polynomial in  $K_1, \dots, K_{n-1}$  with coefficients which are derivatives of  $\mathcal{X}$  of order up to  $n$  evaluated at  $K_0$ . This equation can be written as

$$(L + n\lambda)K_n = TR_n. \quad (5.16)$$

Note that if we proceed by induction in  $n$ , the right-hand side of (5.16) is known and it only remains to determine  $K_n$ .

Eq. (5.16) is readily solvable if  $-n\lambda$  is not in the spectrum of  $L$ , which happens if and only if  $e^{n\lambda}$  is not an eigenvalue of the monodromy matrix  $\Phi_1$ , for  $n \geq 2$ . This is a non-resonance condition for the monodromy matrix. It is readily satisfied if  $e^\lambda$  is the most stable eigenvalue of  $\Phi_1$ , or the most unstable one.

The upshot of the discussion so far is that, provided that  $e^\lambda$  is an eigenvalue of the monodromy matrix but  $e^{n\lambda}$  for  $n \geq 2$  are not, then Eq. (5.2) can be solved in the sense of formal power series. Moreover, once we choose the parameterization of the periodic orbit and an eigenvector  $v$  of  $\Phi_1$  of eigenvalue  $e^\lambda$ , then the first and all higher order terms are uniquely determined.

We can now state the result on invariant manifolds around periodic orbits.

**Theorem 5.4.** *Let  $\mathcal{X}$  be an analytic vector field on  $\mathbb{R}^d$  and assume that it admits a periodic orbit  $\gamma$  of period  $T$ . Let  $\gamma = \gamma(\theta)$ ,  $0 \leq \theta \leq 1$ , be parameterized according to (5.1). Let  $\Phi_1$  be the monodromy matrix associated to  $\gamma$ . That is,  $\Phi_1 = \Phi(1)$ , where  $\Phi_\theta = \Phi(\theta)$  is the fundamental solution of (5.8).*

*Assume that  $\lambda \in \mathbb{R}$  satisfies:*

- (1)  $e^\lambda < 1$  and  $e^\lambda \in \text{Spec}(\Phi_1)$ . Let  $v \in \mathbb{R}^d \setminus \{0\}$  be a solution of  $\Phi_1 v = e^\lambda v$ .
- (2)  $e^{n\lambda} \notin \text{Spec}(\Phi_1)$  for every integer  $n \geq 2$ .

*Then, there exists an analytic two-dimensional manifold invariant under the flow of  $\mathcal{X}$ , containing the periodic orbit  $\gamma$ , and tangent to the space generated by  $\{\dot{\gamma}(\theta), K_1(\theta)\}$  at  $\gamma(\theta)$ ,  $0 \leq \theta \leq 1$ , where  $K_1(\theta) = e^{\mu(1-\theta)}\Phi_\theta\Phi_1^{-1}v$ . In addition, the manifold can be represented as the image of an analytic function  $K$  satisfying (5.2),  $K(\theta, 0) = \gamma(\theta)$  and  $\frac{\partial K}{\partial \sigma}(\theta, 0) = K_1(\theta)$ . Consequently, the motion in the space of parameters is given by (5.3).*

In numerical applications, it is often enough to follow the procedure indicated previously in this section to obtain very approximate representations of the manifold. In the next subsection, we give a complete proof of Theorem 5.4. In particular, we develop estimates for the solutions of equations of the form (5.16) and we study the convergence of the power series, which will give confidence in the numerical analysis. Remark 5.5 explains how the main existence result is an “a posteriori” estimate which justifies the results of numerical calculations.

## 5.2. Convergence of the formal solutions

In this subsection, we show that the solution previously obtained is not just a formal solution, but that it converges and defines an analytic function.

### 5.2.1. Formulation as a fixed point problem

The proof of Theorem 5.4 consists in rewriting (5.2) as a well-posed fixed point problem. As in many of the proofs presented in the paper, we accomplish this by separating a low order part from a high order part of the solution. Here we separate the linear part, that we already have found. Then, once we show that there is a true analytic solution, by the uniqueness of the terms of order bigger or equal than 2, we see that the formal order by order calculation has to produce this analytic solution.

We write

$$\begin{aligned} K(\theta, \sigma) &= K_0(\theta) + K_1(\theta)\sigma + K^>(\theta, \sigma), \\ K^{\leq}(\theta, \sigma) &= K_0(\theta) + K_1(\theta)\sigma. \end{aligned}$$

We assume that  $K_0$  and  $K_1$  are already determined (since we computed them just in the first two steps of the iterative procedure), so that the only unknown is  $K^>$ .

Eq. (5.2) becomes

$$\left( \frac{1}{T} \frac{\partial}{\partial \theta} + \frac{\lambda \sigma}{T} \frac{\partial}{\partial \sigma} \right) (K^{\leq} + K^>) = \mathcal{X} \circ (K^{\leq} + K^>),$$

that we write as

$$\begin{aligned} &\left( \frac{1}{T} \frac{\partial}{\partial \theta} + \frac{\lambda \sigma}{T} \frac{\partial}{\partial \sigma} \right) (K_0 + K_1\sigma + K^>) \\ &= \mathcal{X} \circ K_0 + D\mathcal{X} \circ K_0(K_1\sigma + K^>) + \mathcal{H}(K^>), \end{aligned} \quad (5.17)$$

where  $\mathcal{H}$  is the remainder of Taylor's theorem:

$$\mathcal{H}(K^>) := \mathcal{X} \circ (K_0 + K_1\sigma + K^>) - \mathcal{X} \circ K_0 - D\mathcal{X} \circ K_0(K_1\sigma + K^>). \quad (5.18)$$

Regrouping terms in (5.17) and using (5.5) and (5.6), (5.17) becomes

$$\left( L + \lambda \sigma \frac{\partial}{\partial \sigma} \right) K^> = T\mathcal{H}(K^>). \quad (5.19)$$

The plan of the proof is to show that under the non-resonance conditions of Theorem 5.4, the operator  $L + \lambda \sigma \frac{\partial}{\partial \sigma}$  is boundedly invertible in some appropriately defined spaces. Hence, (5.19) will become

$$K^> = \left( L + \lambda \sigma \frac{\partial}{\partial \sigma} \right)^{-1} T\mathcal{H}(K^>) =: \mathcal{N}(K^>). \quad (5.20)$$

Eq. (5.20) will be analyzed by fixed point methods. We introduce the notation  $\mathcal{N}$  to denote the operator for which we will be seeking fixed points. We will show that the

Lipschitz constant of  $\mathcal{H}$ , and hence of  $\mathcal{N}$ , can be taken to be small if we take  $K_1$  to be small. This is reasonable since  $\mathcal{H}$  is the second-order remainder of the Taylor expansion.

If rather than applying the contraction mapping theorem, we apply to (5.19) the implicit function theorem in Banach spaces, we obtain automatically smooth dependence on parameters for the invariant manifolds.

**Remark 5.5.** Note that one of the conclusions of the fixed point method is that if we obtain an approximate solution of (5.20), that is, a function  $K_{\text{ap}}^>$  such that

$$\|K_{\text{ap}}^> - \mathcal{N}(K_{\text{ap}}^>)\| \leq \delta, \quad (5.21)$$

then there exists a true solution  $K^>$  of the equation such that

$$\|K^> - K_{\text{ap}}^>\| \leq C\delta,$$

where the constant  $C$  will be rather explicit from the proof.

This can be used to justify numerical calculations. A careful numerical implementation of the algorithm discussed at the beginning of the section produces a function  $K_{\text{ap}}^>$  for which  $\delta$  in (5.21) is just the round off error plus the truncation error, which in several practical applications—e.g. [Cap04]—can be made to be a few thousand times the round-off unit. Hence, the proof presented here will ensure that the numerically computed solutions are close to the true one. This procedure is what is usually called “a posteriori estimates” in numerical analysis.

Moreover, we establish that the true solution is close to the computed one in an analytic norm defined below. Hence, the computed solution gives information not only about the location of the stable manifold, but also about its derivatives. This makes possible to discuss bifurcations, tangencies, etc.

### 5.2.2. Norms

In this section we introduce norms that are convenient to carry out the contraction mapping argument.

Let  $\alpha$  and  $\beta$  be positive numbers. We consider the sets

$$\begin{aligned} S_\alpha &= \{\theta \in \mathbb{C}/\mathbb{Z} \mid |\operatorname{Im} \theta| < \alpha\}, \\ D_\beta &= \{\sigma \in \mathbb{C} \mid |\sigma| < \beta\}, \\ U_{\alpha,\beta} &= S_\alpha \times D_\beta. \end{aligned}$$

Note that  $S_\alpha$  can be considered as a complex extension of the torus  $\mathbb{T}^1$ . Functions in  $S_\alpha$  can be identified with functions of period 1 defined in a complex strip.

We consider the space of functions

$$\Gamma_\alpha = \{f : \overline{S_\alpha} \rightarrow \mathbb{C}^d \mid f \text{ continuous and analytic in } S_\alpha\}.$$

We endow  $\Gamma_\alpha$  with the norm  $\|f\|_{\Gamma_\alpha} = \sup_{\theta \in \overline{S_\alpha}} |f(\theta)|$ . As it is well known, this norm makes  $\Gamma_\alpha$  a Banach space.

Given a function  $K$  defined on  $U_{\alpha,\beta}$  which is analytic in both variables, we write it as

$$K(\theta, \sigma) = \sum_{n=0}^{\infty} K_n(\theta) \sigma^n,$$

and we denote by  $H_{\alpha,\beta}$  the space of analytic functions for which

$$\|K\|_{H_{\alpha,\beta}} := \sum_{n=0}^{\infty} \|K_n\|_{\Gamma_\alpha} \beta^n < \infty.$$

It is an easy exercise to check that  $H_{\alpha,\beta}$  is a Banach space with the norm  $\|\cdot\|_{H_{\alpha,\beta}}$ .

We will use the same notation for norms of functions from these domains taking values in other spaces (e.g., matrices). When we consider functions which take values in spaces for which there is a multiplication (e.g., matrix-valued functions and vector-valued functions), the spaces  $\Gamma_\alpha$  and  $H_{\alpha,\beta}$  inherit Banach algebra properties. For example, if  $m$  and  $M$  are  $d \times d$  matrix-valued functions defined in  $S_\alpha$  and  $U_{\alpha,\beta}$ , respectively, and if  $v$  and  $V$  are  $d$ -dimensional vector-valued functions defined in  $S_\alpha$  and  $U_{\alpha,\beta}$ , respectively, then we have

$$\begin{aligned} \|mv\|_{\Gamma_\alpha} &\leq \|m\|_{\Gamma_\alpha} \|v\|_{\Gamma_\alpha}, \\ \|MV\|_{H_{\alpha,\beta}} &\leq \|M\|_{H_{\alpha,\beta}} \|V\|_{H_{\alpha,\beta}}. \end{aligned} \quad (5.22)$$

The first inequality in (5.22) is just that the supremum of the product is less than the product of the suprema. The second inequality is a consequence of the well-known product formula for power series:

$$MV = \sum_{n=0}^{\infty} \sigma^n \sum_{k=0}^n M_k V_{n-k}.$$

Therefore,

$$\|MV\|_{H_{\alpha,\beta}} = \sum_{n=0}^{\infty} \beta^n \left\| \sum_{k=0}^n M_k V_{n-k} \right\|_{\Gamma_\alpha}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} \beta^k \beta^{n-k} \sum_{k=0}^n \|M_k\|_{\Gamma_\alpha} \|V_{n-k}\|_{\Gamma_\alpha} \\
&= \left( \sum_{k=0}^{\infty} \beta^k \|M_k\|_{\Gamma_\alpha} \right) \left( \sum_{i=0}^{\infty} \beta^i \|V_i\|_{\Gamma_\alpha} \right).
\end{aligned}$$

**Remark 5.6.** There are other norms that we could have used in the proof. For example, we could take the suprema on both variables, which in some respects is more natural. The choice we have made is based on the observation that since the argument is based in solving equations for each coefficient in  $\sigma$ , it is natural to use a norm that emphasizes the role of the coefficients in powers of  $\sigma$ . At the same time, the supremum norm in the angle variables makes simpler to obtain estimates for expressions such as (5.11).

The inequality

$$\sup_{(\theta, \sigma) \in U_{\alpha, \beta}} |K(\theta, \sigma)| \leq \|K\|_{H_{\alpha, \beta}} \quad (5.23)$$

is obvious from the triangle inequality. On the other hand, using Cauchy integral formula

$$K_n(\theta) = \frac{1}{2\pi i} \int_{|z|=\beta} z^{-(n+1)} K(\theta, z) dz,$$

we obtain

$$\|K_n\|_{\Gamma_\alpha} \leq \beta^{-n} \sup_{(\theta, \sigma) \in U_{\alpha, \beta}} |K(\theta, \sigma)|. \quad (5.24)$$

This immediately gives

$$\|K\|_{H_{\alpha, \beta-\delta}} \leq \beta \delta^{-1} \sup_{(\theta, \sigma) \in U_{\alpha, \beta}} |K(\theta, \sigma)|. \quad (5.25)$$

Inequalities (5.23) and (5.25) allow us to use supremum to estimate derivatives.

It is also convenient to think of functions in  $H_{\alpha, \beta}$  as analytic functions on  $D_\beta$  with values in the space  $\Gamma_\alpha$ .

Recall that  $K(\theta, \sigma) = K_0(\theta) + K_1(\theta)\sigma + K^>(\theta, \sigma)$  and that we consider  $K$  in the space  $H_{\alpha, \beta}$ . We therefore take the function  $K^>$  in the space

$$H_{\alpha, \beta}^2 = \{K^> \in H_{\alpha, \beta} \mid K^>(\theta, 0) \equiv \partial_\sigma K^>(\theta, 0) \equiv 0\}.$$

Equivalently, we are requiring that the two first coefficients  $(K^>)_0 = (K^>)_1 \equiv 0$  in the expansion of  $K^>$  in powers of  $\sigma$ . We endow  $H_{\alpha, \beta}^2$  with the norm  $\|\cdot\|_{H_{\alpha, \beta}^2} = \|\cdot\|_{H_{\alpha, \beta}}$

inherited from  $H_{\alpha,\beta}$ . Finally, we consider the operators  $\mathcal{H}$  and  $\mathcal{N}$ , defined in (5.18) and (5.20), acting on functions  $K^>$  in the space  $H_{\alpha,\beta}^2$ .

### 5.2.3. Estimates for the linearized equation

We start establishing estimates in the space  $H_{\alpha,\beta}^2$  for solutions of the equation

$$\left(L + \lambda\sigma \frac{\partial}{\partial\sigma}\right)\Psi(\theta, \sigma) = \eta(\theta, \sigma). \quad (5.26)$$

This will be an easy task since (5.26) is equivalent, matching coefficients in  $\sigma^n$ , to

$$(L + n\lambda)\Psi_n(\theta) = \eta_n(\theta), \quad n \geq 2. \quad (5.27)$$

The estimates for each of these coefficients can be readily obtained from Proposition 5.2.

**Lemma 5.7.** Assume that, for some  $\alpha > 0$  and some constants  $C_1$  and  $C_2$ ,

- (1)  $\|\Phi\|_{\Gamma_\alpha} \leq C_1$  and  $\|\Phi^{-1}\|_{\Gamma_\alpha} \leq C_1$ , where  $\Phi_\theta$  is the fundamental solution of the linearized equation.
- (2)  $\mu < 0$  and  $e^\mu$  is not an eigenvalue of the monodromy matrix  $\Phi_1$ .
- (3)  $\|(\Phi_1 - e^\mu \text{Id})^{-1}\| \leq C_2$ .

Then, the solution  $\Delta$  of the linearized equation (5.10) satisfies

$$\|\Delta\|_{\Gamma_\alpha} \leq C \|R\|_{\Gamma_\alpha}, \quad (5.28)$$

where  $C = C_1^2(1 + C_2)$ .

The proof follows from formulas (5.11) and (5.12), using that  $\mu < 0$ .

From now on, we take  $\alpha > 0$  small enough so that  $\gamma = K_0$ ,  $\Phi$  and  $\Phi^{-1}$  all belong to the space  $\Gamma_\alpha$ , as in the previous proposition.

**Lemma 5.8.** Let  $\alpha > 0$  be small as indicated above, and let  $\beta > 0$ . Assume that  $\lambda < 0$  is such that  $e^{n\lambda}$  is not an eigenvalue of the monodromy matrix  $\Phi_1$  for every integer  $n \geq 2$ .

Let  $\eta(\theta, \sigma) = \sum_{n=2}^{\infty} \eta_n(\theta)\sigma^n$  be a function in  $H_{\alpha,\beta}^2$ . Then, there is one and only one  $\Psi \in H_{\alpha,\beta}^2$  solving (5.26). In addition, it satisfies

$$\|\Psi\|_{H_{\alpha,\beta}^2} \leq C \|\eta\|_{H_{\alpha,\beta}^2},$$

for some constant  $C$ .

**Proof.** We use that (5.26) is equivalent to the system of Eq. (5.27),  $n \geq 2$ .

Next, under the assumptions that  $\lambda < 0$  and  $e^{\lambda n}$  is not an eigenvalue of  $\Phi_1$  for  $n \geq 2$ , we can bound  $\|(\Phi_1 - e^{\lambda n} \text{Id})^{-1}\|$  uniformly in  $n$ . Here we have used the non-resonance conditions together with the fact that  $\Phi_1 - e^{\lambda n} \text{Id} \rightarrow \Phi_1$ , which is invertible, as  $n \rightarrow +\infty$ .

Hence, the bounds we obtain applying Lemma 5.7 to each of the coefficients  $\Psi_n$  are uniform in  $n$ . Therefore  $\|\Psi_n\|_{\Gamma_x} \leq C \|\eta_n\|_{\Gamma_x}$  for some constant  $C$  independent of  $n$ , from which the desired result follows.  $\square$

Now we turn to show that the maps  $\mathcal{H}$  and  $\mathcal{N}$  defined in (5.18) and (5.20) are indeed well defined in the space  $H_{\alpha,\beta}^2$ . Moreover, we will show that if  $K_1$  is chosen small enough, we can get a ball centered at 0 mapped by  $\mathcal{N}$  into itself, and on which the Lipschitz constant of  $\mathcal{N}$  is small.

**Proposition 5.9.** *Let  $K_0$  and  $K_1$  be chosen as throughout this section. Let  $\alpha > 0$  be small as indicated above, and let  $\beta, r$  and  $\rho$  be positive numbers. Assume that:*

- (1) *The vector field  $\mathcal{X}$  is analytic and bounded in a domain that includes the complex ball of radius  $r$  around each point  $K_0(\theta)$ ,  $\theta \in S_x$ . Let*

$$a := \sup_{\theta \in S_x} \sup_{|z - K_0(\theta)| \leq r} |\mathcal{X}(z)|.$$

- (2)  $\|K_1(\theta)\sigma\|_{H_{\alpha,\beta}} \leq \rho$ .

- (3)  $\rho \leq r/4$ .

Then

- (a) *The map  $K^> \in B_\rho(0) \subset H_{\alpha,\beta}^2 \mapsto \mathcal{X}(K_0 + K_1\sigma + K^>) \in H_{\alpha,\beta}$  is well defined, where  $B_\rho(0)$  is the ball in  $H_{\alpha,\beta}^2$  of radius  $\rho$  centered at the origin.*
- (b) *If  $K^> \in B_\rho(0) \subset H_{\alpha,\beta}^2$  then*

$$\mathcal{H}(K^>) = \mathcal{X}(K_0 + K_1\sigma + K^>) - \mathcal{X} \circ K_0 - D\mathcal{X} \circ K_0(K_1\sigma + K^>) \quad (5.29)$$

*belongs to  $H_{\alpha,\beta}^2$ .*

- (c) *For every  $K^>$  and  $\tilde{K}^>$  in  $B_\rho(0) \subset H_{\alpha,\beta}^2$ , we have*

$$\|\mathcal{H}(K^>)\|_{H_{\alpha,\beta}^2} \leq 8ar^{-2}\rho^2, \quad (5.30)$$

$$\|\mathcal{H}(K^>) - \mathcal{H}(\tilde{K}^>)\|_{H_{\alpha,\beta}^2} \leq 12ar^{-2}\rho\|K^> - \tilde{K}^>\|_{H_{\alpha,\beta}^2}. \quad (5.31)$$



**Proof.** By the analyticity properties of  $\mathcal{X}$  and hypothesis (1), we see that if

$$\mathcal{X}(K_0(\theta) + z) = \sum_{n \geq 0} \mathcal{X}_n(\theta) z^n,$$

we then have

$$|\mathcal{X}_n(\theta)| \leq ar^{-n}.$$

This is proved using Cauchy integral formula as in (5.24) before. Since the above inequality is true for any  $\theta \in S_x$ , we have  $\|\mathcal{X}_n\|_{\Gamma_x} \leq ar^{-n}$ .

Using the Banach algebra properties of our spaces of functions, we see that

$$\mathcal{X}(K_0(\theta) + K_1(\theta)\sigma + K^>(\theta, \sigma)) = \sum_{n=0}^{\infty} \mathcal{X}_n(\theta) (K_1(\theta)\sigma + K^>(\theta, \sigma))^n \quad (5.32)$$

is well defined and converges uniformly, for  $K^> \in B_\rho(0)$ . Indeed, the  $H_{x,\beta}$ -norm of each term in the series is bounded by  $\|\mathcal{X}_n\|_{H_{x,\beta}} (\|K_1(\theta)\sigma\|_{H_{x,\beta}} + \|K^>\|_{H_{x,\beta}})^n \leq ar^{-n}(2\rho)^n$ , which converges by assumption (3).

Next, from (5.32) and definition (5.29) of  $\mathcal{H}$ , we see that

$$\mathcal{H}(K^>)(\theta, \sigma) = \sum_{n=2}^{\infty} \mathcal{X}_n(\theta) (K_1(\theta)\sigma + K^>(\theta, \sigma))^n.$$

The same argument as before establishes that this series converges, and hence  $\mathcal{H}(K^>)$  is well defined in  $H_{x,\beta}^2$ . Moreover, using assumptions (2) and (3), we obtain

$$\begin{aligned} \|\mathcal{H}(K^>)\|_{H_{x,\beta}^2} &\leq \sum_{n=2}^{\infty} \|\mathcal{X}_n\|_{H_{x,\beta}} \left( \|K_1(\theta)\sigma\|_{H_{x,\beta}} + \|K^>\|_{H_{x,\beta}} \right)^n \\ &\leq \sum_{n=2}^{\infty} ar^{-n}(2\rho)^n = a(2r^{-1}\rho)^2 \frac{1}{1 - 2r^{-1}\rho} \\ &\leq 8a(r^{-1}\rho)^2, \end{aligned}$$

as claimed in (5.30).

Finally, we also have

$$\begin{aligned} &\|\mathcal{H}(K^>) - \mathcal{H}(\tilde{K}^>)\|_{H_{x,\beta}^2} \\ &\leq \sum_{n=2}^{\infty} \|\mathcal{X}_n\|_{H_{x,\beta}} \left\| (K_1(\theta)\sigma + K^>(\theta, \sigma))^n - (K_1(\theta)\sigma + \tilde{K}^>(\theta, \sigma))^n \right\|_{H_{x,\beta}} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=2}^{\infty} ar^{-n} \|K^> - \tilde{K}^>\|_{H_{x,\beta}^2} n(2\rho)^{n-1} \\ &= ar^{-1} \|K^> - \tilde{K}^>\|_{H_{x,\beta}^2} \sum_{n=2}^{\infty} n(2r^{-1}\rho)^{n-1} \leq 12ar^{-2}\rho \|K^> - \tilde{K}^>\|_{H_{x,\beta}^2}. \end{aligned}$$

This establishes (5.31).  $\square$

Note that assumptions (2) and (3) in the previous proposition can be accomplished either by taking  $K_1$  small enough (recall that  $K_1$  is defined up to a multiplicative constant), or by taking  $\beta$  small enough.

Since  $\mathcal{N} = (L + \lambda\sigma \frac{\partial}{\partial\sigma})^{-1}T\mathcal{H}$ , we see from estimates (5.30), (5.31) and Lemma 5.8 that if we choose  $\rho$  small enough, we obtain a ball that gets mapped into itself by  $\mathcal{N}$  and on which the map  $\mathcal{N}$  is a contraction. Therefore,  $\mathcal{N}$  has a unique fixed point in such ball. This finishes the proof of the result of this section.

**Remark 5.10.** In [Mey75] one can find that the operator  $\mathcal{N}$  is actually analytic in the indicated spaces. Hence, we could use the implicit function theorem and obtain automatically smooth dependence on parameters.

**Remark 5.11.** When one considers a discrete time dynamical system (i.e., a map) whose inverse is entire (e.g. polynomial), we argued in Remark 4.5 that the solution  $K$  is entire.

In contrast, the solutions of polynomial differential equations usually are not entire, and very often they present essential singularities. Hence, when working with differential equations, one should not expect the coefficients to decay fast. Choosing the  $b$  as indicated in Remark 5.3 is quite important for numerical applications.

## 6. A $C^0$ invariant stable manifold theorem

In this section we prove a version of the stable manifold theorem. It is not optimal in terms of the regularity obtained, but its proof is simple and obtains differentiability with respect to parameters. In contrast to other sections in this paper, we formulate the results for maps in general Banach spaces. We also call attention to some rather subtle technicalities such as the fact that, in infinite-dimensional spaces, the existence of smooth cut-off functions cannot be taken for granted.

To state it, we use the following terminology. We say that a map is  $C_u^1$  if it is of class  $C^1$  and has uniformly continuous derivative. Recall that for a function defined in a finite-dimensional space, if the function is  $C^1$  in a neighborhood of a point then it is  $C_u^1$  in a smaller neighborhood—since continuous functions in compact sets are uniformly continuous. Recall also that if a function is  $C^{1+\varepsilon}$  for some  $\varepsilon > 0$ , then it is  $C_u^1$ , even in infinite dimensions.

**Theorem 6.1.** *Let  $X$  be a Banach space, and  $F:U \subset X \rightarrow X$  be a  $C^1$  map in a neighborhood  $U$  of 0, with  $F(0) = 0$ . Let  $A := DF(0)$ . Assume:*

- (1)  *$A$  is an invertible operator.*
- (2) *There exists a decomposition*

$$X = X_s \oplus X_u$$

*such that:*

- (2.1) *It is invariant under  $A$ . That is,*

$$AX_s \subset X_s, \quad AX_u \subset X_u.$$

- (2.2) *Let  $A_s := \pi_s A|_{X_s}$  and  $A_u := \pi_u A|_{X_u}$ , where  $\pi_s$  and  $\pi_u$  are the projections onto  $X_s$  and  $X_u$ , respectively. Suppose that  $\|A_s\| < 1$  and  $\|A_u^{-1}\| < 1$ .*

- (3) *If  $X$  is infinite dimensional, assume that  $X$  admits smooth cut-off functions, and that  $F$  is  $C_u^1(U)$  (i.e.,  $DF$  is uniformly continuous in  $U$ ).*

*Then, there exists a continuous map  $K:U_1 \subset X_s \rightarrow X$ , where  $U_1$  is a neighborhood of 0, such that*

- (a)  $K(0) = 0$ .
- (b)  $F \circ K = K \circ A_s$  in  $U_1$ .

*Moreover, assume that  $F_\lambda$  is a  $C^1$  family of  $C_u^1$  maps (i.e., the map  $\lambda \in V \mapsto F_\lambda \in C_u^1$ , where  $V$  is a neighborhood of 0 in  $\Lambda$ , is  $C^1$  when the maps  $F_\lambda$  are given the  $C^1$  topology) with  $F_\lambda(0) = 0$ , and that  $F_0$  satisfies the hypotheses above. Then, for  $\lambda$  small enough, there exists a continuous map  $K_\lambda$  satisfying  $K_\lambda(0) = 0$  and*

$$F_\lambda \circ K_\lambda = K_\lambda \circ A_{0,s} \quad \text{in a neighborhood of the origin,}$$

*where  $A_{0,s} = \pi_s A_0|_{X_s}$  and  $A_0 = DF_0(0)$ . In addition, the map  $\lambda \mapsto K_\lambda$  is  $C^1$  in a neighborhood of 0 when the maps  $K_\lambda$  are given the  $C^0$  topology.*

**Remark 6.2.** The hypothesis on cut-off functions, made in (3), means that there exists a  $C^\infty$  function  $\xi: X \rightarrow \mathbb{R}$  which is identically 1 in the unit ball centered at 0 and identically 0 outside of the ball of radius 2 centered at 0. Such function always exists if  $X$  is finite dimensional or a Hilbert space. It suffices to take  $\Psi(|x|)$ , where  $\Psi$  is a  $C^\infty$  real valued function over the reals with the indicated properties and  $|\cdot|$  is a norm which comes from a scalar product.

Perhaps surprisingly, the existence of smooth cut-off functions is not true for arbitrary Banach spaces. For example,  $C^0[0, 1]$  does not admit a  $C^2$  cut-off function. We refer to [DGZ93].

The previous result, Theorem 6.1, is far from optimal in several respects. For example, the regularity can be improved, and the existence of cut-off functions can be eliminated

(see the following remark). Nevertheless, we point out the naturalness and the speed of the proof.

**Remark 6.3.** For some  $\rho > 0$  small enough, the local stable invariant manifold  $K(B_\rho(0))$  can be characterized by the following dynamical property:

$$K(B_\rho(0)) \cap V = \{x \in V \mid F^{(i)}(x) \in V \text{ for all } i \geq 0\}$$

for every sufficiently small neighborhood  $V$  of the origin; see [PdM82]. This fact automatically implies the uniqueness of the local invariant manifold. The above characterization (and some extra work) also allows to establish that  $K$  is indeed differentiable at 0 and that  $DK(0) = (\text{Id}, 0)$ . That is,  $K(B_\rho(0))$  is tangent to  $X_s$  at the origin. We refer to [PdM82] for all these matters.

**Proof of Theorem 6.1.** We write  $A := DF(0)$  and  $N(x) := F(x) - Ax$ . A preliminary reduction, standard in the field, is to consider the function

$$F^\delta(x) := Ax + N^\delta(x), \quad \text{where } N^\delta(x) := \zeta(x) \frac{N(\delta x)}{\delta}$$

for  $\delta > 0$  small enough. Here  $\zeta: X \rightarrow \mathbb{R}$  is a smooth cut-off function ( $\zeta \equiv 1$  in the unit ball centered at 0 and  $\zeta \equiv 0$  outside of the ball of radius 2 centered at 0). Even if  $N$  is only defined in a neighborhood of the origin, for  $\delta$  small enough we may consider the non-linearity  $N^\delta$  to be defined and be  $C_u^1$  in the whole  $X$ , since  $\zeta$  has bounded support. We use here that, in finite dimensions, every  $C^1$  function in a neighborhood of 0 is automatically  $C_u^1$  in a smaller neighborhood (since continuous functions in compact sets are uniformly continuous).

It is important to note that if  $W^\delta$  is a manifold through 0 invariant under  $F^\delta$ , then  $\delta W^\delta$  is invariant under  $F$  in a neighborhood of 0. It suffices therefore to find an invariant manifold for  $F^\delta$  for  $\delta$  small enough. This will be accomplished using the implicit function theorem. For this, we consider the non-linearity  $N^\delta$  as belonging to the following Banach space.

We work in the space  $C_{0,u}^1$  of bounded  $C^1$  maps  $M: X \rightarrow X$  with bounded and uniformly continuous derivative  $DM$  in the whole  $X$ , and such that  $M(0) = DM(0) = 0$ . We equip this space with the standard  $C^1$  norm.

Note that  $N^\delta \in C_{0,u}^1$  for all  $\delta$  small enough. Using that  $N(0) = DN(0) = 0$ , it is easy to check that, by choosing  $\delta$  small enough, we can assume that  $\|N^\delta\|_{C^1}$  is as small as we want.

Abusing of notation, we work with maps  $F = A + N$  with  $N \in C_{0,u}^1$ —where in reality, all what follows is applied to  $F^\delta = A + N^\delta$ , for which  $N^\delta$  does truly belong to  $C_{0,u}^1$ .

We write the parameterization as

$$K = (\text{Id}, 0) + K^>,$$

and look for  $K^>$  in the Banach space  $C_0^0 = C_0^0(X_s; X)$  of bounded continuous maps  $K^> : X_s \rightarrow X$  with  $K^>(0) = 0$ . We have

$$\begin{aligned} F \circ K - K \circ A_s &= (A + N) \circ ((\text{Id}, 0) + K^>) - ((\text{Id}, 0) + K^>) \circ A_s \\ &= A \circ K^> + N \circ ((\text{Id}, 0) + K^>) - K^> \circ A_s. \end{aligned}$$

Theorem 6.1 follows by application of the implicit function theorem in Banach spaces to the operator

$$\mathcal{T}(N, K^>) := A \circ K^> + N \circ ((\text{Id}, 0) + K^>) - K^> \circ A_s,$$

considered as an operator from  $C_{0,u}^1 \times C_0^0$  to  $C_0^0$ .

We want to solve  $\mathcal{T}(N, K^>) = 0$  and obtain  $K^>$  as a function of  $N$ , for  $N$  near 0.

Note that  $\mathcal{T}(0, 0) = 0$ —this corresponds to the linear map  $F = A$ , for which the invariant manifold is  $X_s$ . Recall also that, by taking  $\delta$  small, we may assume that  $N = N^\delta$  is as small in  $C_{0,u}^1$  as needed.

It is easy to verify (see [dlO99]) that the operator  $\mathcal{T}$  is  $C^1$ , and that

$$D_2\mathcal{T}(N, K^>)\Delta = A\Delta + DN \circ ((\text{Id}, 0) + K^>)\Delta - \Delta \circ A_s.$$

It suffices to verify that the expression above satisfies the definition of derivative. The verification of the definition of derivative is where we use that  $DN$  is uniformly continuous. More details on the verification and examples that show that uniform continuity of  $DN$  is needed to get differentiability of  $\mathcal{T}$  can be found in [dlO99].

To complete the proof, we only need to show that the operator

$$S := D_2\mathcal{T}(0, 0),$$

given by

$$S\Delta = A\Delta - \Delta \circ A_s,$$

is invertible from  $C_0^0$  to  $C_0^0$ . This amounts to, given  $\eta \in C_0^0$ , find  $\Delta \in C_0^0$  such that

$$A\Delta - \Delta \circ A_s = \eta \tag{6.1}$$

and show that  $\|\Delta_u\|_{C^0} \leq C\|\eta\|_{C^0}$ .

Taking components along  $X_s$  and  $X_u$ , and using the invariance of these subspaces, this is equivalent to

$$A_s\Delta_s - \Delta_s \circ A_s = \eta_s,$$

$$A_u\Delta_u - \Delta_u \circ A_s = \eta_u,$$

which in turn is equivalent to

$$\begin{aligned}\Delta_s &= A_s \Delta_s \circ A_s^{-1} - \eta_s \circ A_s^{-1}, \\ \Delta_u &= A_u^{-1} \Delta_u \circ A_s + A_u^{-1} \eta_u.\end{aligned}\tag{6.2}$$

Eqs. (6.2) can be readily solved as

$$\begin{aligned}\Delta_s &= - \sum_{i=0}^{\infty} A_s^i \eta_s \circ A_s^{-i-1}, \\ \Delta_u &= \sum_{i=0}^{\infty} A_u^{-(i+1)} \eta_u \circ A_s^i.\end{aligned}\tag{6.3}$$

The right-hand sides of (6.3) define a bounded operator on  $C_0^0$ . Indeed, note that the  $C^0$  norms of the general terms are bounded by convergent geometric series, thanks to hypothesis (2.2), and we obtain  $\|\Delta_s\|_{C^0} \leq \sum_{i=0}^{\infty} \|A_s\|^i \|\eta_s\|_{C^0}$ ,  $\|\Delta_u\|_{C^0} \leq \sum_{i=0}^{\infty} \|A_u^{-1}\|^i \|\eta_u\|_{C^0}$ , thus

$$\|\Delta_u\|_{C^0} = \|\mathcal{S}^{-1}\eta\|_{C^0} \leq C\|\eta\|_{C^0}.$$

Finally, we prove the result about dependence on parameters. If  $F_\lambda$  is a family of maps as in the statement, we consider

$$F_\lambda^\delta(x) := A_0 x + \zeta(x) \frac{(F_\lambda - A_0)(\delta x)}{\delta}.$$

Note that

$$N_\lambda^\delta(x) := \zeta(x) \frac{(F_\lambda - A_0)(\delta x)}{\delta}$$

is as small as we want in  $C_{0,u}^1$ , if  $\delta$  and  $\lambda$  are small. Note also that, given  $\delta_0$  sufficiently small, the map  $\lambda \in V \mapsto N_\lambda^{\delta_0} \in C_{0,u}^1$  is  $C^1$ . Hence, composing this map with the function provided by the implicit function theorem, we conclude the result for the parameter dependence.  $\square$

Eq. (6.1) is called a cohomology equation. In Appendix A we describe some basic facts about the solvability of this type of equations.

## Part II

### 7. $C^r$ One-dimensional invariant manifolds

In this section, we discuss the modifications needed in Section 4 to cover the case when the map  $F$  is not analytic but just differentiable. The main result of this section is:

**Theorem 7.1.** *Let  $F: U \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a  $C^{r+1}$  map in a neighborhood  $U$  of the origin, with  $F(0) = 0$ . Denote  $A = DF(0)$ . Let  $\lambda \in \mathbb{R}$  be an eigenvalue of  $A$  and let  $v \in \mathbb{R}^d \setminus \{0\}$  satisfy  $Av = \lambda v$ . Assume:*

- (1)  $A$  is invertible.
- (2)  $0 < |\lambda| < 1$ .

Denote by  $L \geq 1$  an integer large enough such that  $|\lambda|^{L+1} \|A^{-1}\| < 1$ .

- (3)  $\lambda^n \notin \text{Spec } A$  for  $n = 2, \dots, L$ .
- (4)  $L + 1 \leq r$ .

Then, there exists a  $C^r$  map  $K: U_1 \subset \mathbb{R} \rightarrow \mathbb{R}^d$ , where  $U_1$  is a neighborhood of 0, such that

$$F \circ K = K \circ \lambda \quad (7.1)$$

in  $U_1$ ,  $K(0) = 0$  and  $K'(0) = v$ . Hence, the range of  $K$  is a  $C^r$  manifold invariant under  $F$  and tangent to  $v$  at the origin.

Moreover, if  $\hat{K}$  is another  $C^{L+1}$  solution of (7.1) in a neighborhood of the origin, with  $\hat{K}(0) = 0$  and  $\hat{K}'(0) = \beta K'(0)$  for some  $\beta \in \mathbb{R}$ , then  $\hat{K}(t) = K(\beta t)$  for  $t$  small enough.

Note that the transformations needed to go from (4.2) to (4.6) are purely algebraic manipulations and, therefore, they work exactly in the same way as in the case of Section 4, where  $F$  was analytic. However, the spaces in which we define  $\mathcal{T}$  are different. Here we take

$$\Gamma = \{K^> \in C^r(\bar{B}_1(0)) \mid K^>(0) = 0, \quad DK^>(0) = 0\}$$

endowed with the  $C^r$  topology. We recall that

$$\mathcal{T}(K_1, K^>)(z) = (SK^>)(z) + N(K_1 z + K^>(z)),$$

where

$$(\mathcal{S}\Delta)(z) = A\Delta(z) - \Delta(\lambda z).$$

The following result is well known (see for instance [AMR83,dILO99]).

**Proposition 7.2.** *Assume that  $F \in C^{r+1}(\bar{B}_1(0))$ . Then,*

- (1) *The operator  $\mathcal{T}: V \subset \mathbb{R}^d \times \Gamma \rightarrow \Gamma$  is  $C^1$  in a neighborhood  $V$  of  $(0, 0)$ .*
- (2)  $D_2\mathcal{T}(0, 0) = \mathcal{S}$ .

From this, to apply the implicit function theorem to equation  $\mathcal{T} = 0$ , it only remains to establish the invertibility of  $\mathcal{S}$ .

The following is the crucial point in the finite-differentiability theory. The basic idea is that we can invert  $\mathcal{S}$  in spaces of functions that vanish at the origin to high enough order, and that the lower order terms can be handled by the non-resonance assumptions. This remark played an important role in [Ste57] and, more explicitly in [BdILW96,CFdILO3a]. It will also play an important role in our treatment of non-resonant invariant manifolds in future sections.

The following lemma is a particular case of Lemma 5 in [BdILW96].

**Lemma 7.3.** *Under the hypotheses of Theorem 7.1,  $\mathcal{S}$  is boundedly invertible from  $\Gamma$  to  $\Gamma$ .*

**Proof.** The goal is to provide a  $C^r$  solution of the same equation that was considered in the analytic case in Lemma 4.7. In that case, we produced a solution by just equating coefficients. In the case of finitely differentiable functions, it is more expedient to express the solution in terms of highly iterated functions (see formula (7.5)).

We consider the equation

$$\mathcal{S}\Delta = \eta \tag{7.2}$$

for a given  $\eta \in \Gamma$ . We write

$$\eta(t) = \sum_{n=2}^L \eta_n t^n + \tilde{\eta}(t),$$

where  $D^i \tilde{\eta}(0) = 0$  for  $i = 0, \dots, L$ . Clearly we have that  $|\eta_n| \leq \frac{1}{n!} \|\eta\|_{C^r}$  and  $\|\tilde{\eta}\|_{C^r} \leq C \|\eta\|_{C^r}$ .

We seek  $\Delta$  expressed in a similar way, i.e.  $\Delta(t) = \sum_{n=2}^L \Delta_n t^n + \tilde{\Delta}(t)$ . Equating powers of  $t$ , it is clear that (7.2) is equivalent to

$$(A - \lambda^n)\Delta_n = \eta_n, \quad n = 2, \dots, L \tag{7.3}$$

and

$$A\tilde{\Delta}(t) - \tilde{\Delta}(\lambda t) = \tilde{\eta}(t). \tag{7.4}$$



Eq. (7.3) can be solved by the non-resonance assumption (3). Since  $(A - \lambda^n)^{-1}$  are bounded for  $n = 2, \dots, L$ , we have  $\|\sum_{n=2}^L \Delta_n t^n\|_{C^r} \leq C\|\eta\|_{C^r}$ .

Next, we claim that the solution of (7.4) is given by

$$\tilde{\Delta}(t) = \sum_{i=0}^{\infty} A^{-i-1} \tilde{\eta}(\lambda^i t). \quad (7.5)$$

First, by Taylor's theorem we have  $|\tilde{\eta}(t)| \leq \|\tilde{\eta}\|_{C^{L+1}} |t|^{L+1} / (L+1)! \leq C\|\eta\|_{C^r} |t|^{L+1}$  and hence

$$\begin{aligned} |A^{-i-1} \tilde{\eta}(\lambda^i t)| &\leq C \|A^{-1}\|^{i+1} |\lambda|^{i(L+1)} |t|^{L+1} \|\eta\|_{C^r} \\ &\leq C(|\lambda|^{L+1} \|A^{-1}\|)^i \|\eta\|_{C^r}. \end{aligned}$$

Therefore, the series in (7.5) converges uniformly on  $\{t \in \mathbb{R} \mid |t| \leq 1\}$  by Weierstrass  $M$ -test.

Moreover, the uniform convergence shows that (7.5) solves Eq. (7.4), since it can be substituted and the terms rearranged to show that indeed solves the equation.

Next, we claim that the series obtained taking derivatives term by term up to order  $r$  in (7.5) also converge uniformly, and that the  $C^0$  norm of the corresponding sum can be bounded by  $C\|\eta\|_{C^r}$ . Indeed, again by Taylor's theorem

$$|D^j \tilde{\eta}(t)| \leq C\|\eta\|_{C^r} |t|^{(L+1-j)_+}, \quad 0 \leq j \leq r,$$

where  $(\cdot)_+ = \max(0, \cdot)$ . Therefore,

$$\begin{aligned} \|D^j A^{-i-1} \tilde{\eta}(\lambda^i t)\|_{C^0(\bar{B}_1(0))} &= \|A^{-i-1} D^j \tilde{\eta}(\lambda^i t) \lambda^{ij}\|_{C^0(\bar{B}_1(0))} \\ &\leq C \|A^{-1}\|^i \|\eta\|_{C^r} |\lambda|^{i(L+1-j)_+} \lambda^{ij} \\ &\leq C(|\lambda|^{L+1} \|A^{-1}\|)^i \|\eta\|_{C^r}, \end{aligned}$$

because  $i(L+1-j)_+ + ij \geq i(L+1)$ . This proves that series (7.5) defines a  $C^r$  function  $\tilde{\Delta}$  and that  $\|\tilde{\Delta}\|_{C^r} \leq C\|\eta\|_{C^r}$ . From these estimates we get  $\|\Delta\|_{C^r} \leq C\|\eta\|_{C^r}$ .  $\square$

The previous results lead rather immediately to the following.

**Proof of Theorem 7.1.** Proposition 7.2 and Lemma 7.3 show that, under the hypotheses of the theorem we can apply the implicit function theorem to the operator  $\mathcal{T}$  and show that for small enough  $K_1$ , we can find  $K^* \in \Gamma$  in such a way that (7.1) is satisfied.

The claimed uniqueness result follows from the simple observation that, if we had  $\widehat{K}$ ,  $K$  as indicated, then  $\widehat{K}(\rho t)$ ,  $K(\rho t)$  for  $\rho$  small enough would satisfy also the hypotheses and would be covered by the uniqueness conclusions of the implicit function theorem applied with  $r = L + 1$ . We then conclude that  $\widehat{K}(\rho t) = K(\beta \rho t)$  in the unit ball, from which the result follows immediately.  $\square$

**Remark 7.4.** By the Sternberg theorem [Ste57] in one dimension, any  $C^\ell$ ,  $\ell \geq 2$ , one-dimensional invariant manifold tangent to a contracting eigenvector has a dynamics which is  $C^\ell$  conjugate to the linear one. Hence any  $C^\ell$ ,  $\ell \geq 2$ , invariant manifold tangent to  $v$  can be obtained by the parameterization method.

The uniqueness statement of Theorem 7.1 establishes that  $C^{L+1}$  manifolds tangent to  $v$  are unique.

**Remark 7.5.** The regularity obtained for the manifolds is not optimal. In the paper [CFdIL03a] it is shown by a careful analysis that, under the same hypothesis as Theorem 7.1, one can obtain that  $K \in C^{r+1}$  (the proof that  $K \in C^{r+\text{Lip}}$  is considerably easier).

**Remark 7.6.** The proof presented here works in the case that  $F$  is just assumed to be a diffeomorphism on a Banach space  $X$ . The only difference is that, for infinite-dimensional Banach spaces to have that the composition  $K^> \mapsto N \circ (K_1 + K^>)$  is  $C^1$  acting from  $C^r$  to  $C^r$  we need to assume  $F \in C^{r+1+\varepsilon}$  (see [dILO99] for a proof and for examples that show that  $C^{r+1}$  is not enough). The reason for the need of an extra  $\varepsilon$  is that in Banach spaces continuity on a ball does not imply uniform continuity.

**Remark 7.7.** Note that the condition  $|\lambda|^{L+1} \|A^{-1}\| < 1$  in Theorem 7.1 can be substituted by a condition which only involves the spectrum of  $A$ . This is accomplished by using the device of adapted norms. See Appendix A of [CFdIL03a].

**Remark 7.8.** Again, we note that the use of the implicit function theorem has as a corollary the smooth dependence of the manifold with respect to the parameters of the problem. We refer to [CFdIL03b] for rather sharp results along these lines.

## 8. A $C^0$ slow manifold theorem

In this section we deal with slow manifolds. They are the invariant manifolds associated to sets of eigenvalues contained in open rings of  $\mathbb{C}$  of radii  $r_-$  and 1. These manifolds catch the dynamics of orbits which tend to the fixed point at the slowest rate. The next result gives sufficient conditions for the existence of these manifolds, essentially based on the spectrum of the linearized map.

**Theorem 8.1.** *Let  $X$  be a Banach space,  $U$  an open subset of  $X$ ,  $0 \in U$ , and  $F : U \rightarrow X$  be a  $C^1$  map such that  $F(0) = 0$ . Let  $A = DF(0)$ . Assume the following hypotheses:*

- (1)  *$F$  is  $C^{1+\varepsilon}(U)$  for some  $0 < \varepsilon < 1$ . If  $X$  is finite dimensional, it suffices to assume that  $F$  is  $C^1$ .*

(2) *There exists a decomposition*

$$X = X_1 \oplus X_2,$$

*which is invariant under A. That is,*

$$AX_1 \subset X_1, \quad AX_2 \subset X_2.$$

(3) *Let  $A_1 = A|_{X_1}$  and  $A_2 = A|_{X_2}$ . Assume that  $A_1$  is invertible,  $\|A_1\| < 1$ , and that there exist  $\nu, \mu \in (0, \varepsilon]$  such that*

$$\|A_1^{-1}\| \|A_1\|^{1+\nu} < 1, \quad \|A_1^{-1}\|^{1+\mu} \|A_2\| < 1.$$

*Then, there exists a continuous map  $K : U_1 \subset X_1 \rightarrow X$  such that*

- (a)  $F \circ K = K \circ A_1$  *in a neighborhood of the origin.*
- (b)  $K(0) = 0$ ,  $K$  *is differentiable at 0 and*  $DK(0) = (\text{Id}, 0)$ .

**Remark 8.2.** By the method of the adapted norms (see Appendix A of the article [CFdIL03a]), condition (3) is satisfied for a norm equivalent to the original one if

$$\rho(A_1^{-1})(\rho(A_1))^{1+\nu} < 1, \quad (\rho(A_1^{-1}))^{1+\mu} \rho(A_2) < 1 \quad (8.1)$$

hold. We recall that  $\rho(A_i)$  is the spectral radius of  $A_i$ .

The following is the typical situation that we have in mind in which (8.1) is satisfied. Assume that  $r_2 \leq R_2 < r_1 \leq R_1 < 1$  and that  $r_1 \leq \text{Spec}(A_1) \leq R_1$  and  $r_2 \leq \text{Spec}(A_2) \leq R_2$ . Then, we can always find  $\mu \in (0, \varepsilon]$  satisfying the second condition of (8.1), but the first condition in (8.1) gives a restriction on the admissible  $A_1$ . The simplest situation where this condition is fulfilled for some  $\nu \in (0, \varepsilon]$  is when  $X_1$  is one dimensional.

**Proof of Theorem 8.1.** We write

$$F(x) = Ax + N(x) = (A_1x_1 + N_1(x), A_2x_2 + N_2(x)),$$

with  $x = x_1 + x_2 \in X_1 \oplus X_2$ . As in Section 6 we scale the map to have it defined on the ball  $\bar{B}_2(0) \subset X$  and to have  $\|N\|_{C^1}$  sufficiently small.

We look for  $K$  in the form

$$K = (\text{Id}, 0) + K^>,$$

where  $K^> = (K_1^>, K_2^>)$  with  $|K_1^>(x)|/|x|^{1+\nu}$  and  $|K_2^>(x)|/|x|^{1+\mu}$  bounded, and such that

$$F \circ K = K \circ A_1 \quad \text{in } B_1(0) \subset X_1. \quad (8.2)$$

Here, and from now on,  $x$  denotes the variable in  $X_1$ .

Since we do not expect to have a unique solution for Eq. (8.2), we look for an operator  $\mathcal{T}$  such that the equation  $\mathcal{T}(N, K^>) = 0$  has a unique solution and its solution is also a solution of (8.2). We will find the solution of  $\mathcal{T}(N, K^>) = 0$  using the implicit function theorem.

Equation  $F \circ K = K \circ A_1$  can be rewritten in the form

$$\begin{aligned} A_1 K_1^> + N_1 \circ K - K_1^> \circ A_1 &= 0, \\ A_2 K_2^> + N_2 \circ K - K_2^> \circ A_1 &= 0. \end{aligned}$$

We introduce the linear map

$$\mathcal{S}(K^>) = (A_1 K_1^> - K_1^> \circ A_1, A_2 K_2^> - K_2^> \circ A_1). \quad (8.3)$$

We will see that  $\mathcal{S}$  has a bounded right inverse  $\mathcal{S}^{-1}$  in a space  $\Gamma$  defined below. It is clear that if  $K^>$  is a solution of

$$K^> + \mathcal{S}^{-1}N \circ ((\text{Id}, 0) + K^>) = 0,$$

then, it is also a solution of (8.2).

Let

$$\|K^>\|_\Gamma = \max \left( \sup_{x \in \bar{B}_1(0)} |K_1^>(x)|/|x|^{1+\nu}, \sup_{x \in \bar{B}_1(0)} |K_2^>(x)|/|x|^{1+\mu} \right)$$

and

$$\Gamma = \{K^> : \bar{B}_1(0) \subset X_1 \longrightarrow X \mid K^> \text{ is continuous, } \|K^>\|_\Gamma < \infty\}.$$

**Lemma 8.3.**  $\mathcal{S} : \Gamma \longrightarrow \Gamma$  has a bounded right inverse.

**Remark 8.4.** The specific construction of the right inverse determines which solution we will finally obtain. The lack of uniqueness of solutions of (8.2) is reflected by the degree of freedom we have in constructing  $\mathcal{S}^{-1}$ .

**Proof of Lemma 8.3.** Let  $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$  be a  $C^1$  cut-off function such that  $0 \leq \xi(t) \leq 1$ ,  $\xi(t) = 1$  if  $0 \leq t \leq 1$ , and  $\xi(t) = 0$  if  $t \geq 2$ .

To prove the existence of a right inverse  $\mathcal{S}^{-1}$ , for any given  $\eta = (\eta_1, \eta_2) \in \Gamma$  we have to find  $\Delta = (\Delta_1, \Delta_2) \in \Gamma$  such that

$$\mathcal{S}\Delta = \eta.$$

The formulas

$$\Delta_1 = \sum_{i=0}^{\infty} A_1^{-(i+1)} \eta_1 \circ A_1^i, \quad (8.4)$$

$$\Delta_2 = - \sum_{i=0}^{\infty} A_2^i \eta_2 \circ A_1^{-(i+1)}, \quad (8.5)$$

give a formal solution. It will be a true solution provided the series converge uniformly and  $\eta_2$  is globally defined in order that the formula for  $\Delta_2$  makes sense. To deal with the second difficulty we only have to extend  $\eta_2$  in a continuous way, which is always possible in a Banach space as follows. We substitute  $\eta_2$  by the extension  $\tilde{\eta}_2$  defined by

$$\begin{aligned} \tilde{\eta}_2(x) &= \eta_2(x), \quad \text{if } |x| < 1, \\ \tilde{\eta}_2(x) &= \zeta(|x|)\eta_2(x/|x|), \quad \text{if } |x| \geq 1. \end{aligned}$$

Note that if  $|A_1^{-(i+1)}x| < 1$  then  $|\tilde{\eta}_2(A_1^{-(i+1)}x)| \leq \|\eta\|_{\Gamma}(|A_1^{-(i+1)}x|)^{1+\mu}$ . Also, if  $|A_1^{-(i+1)}x| \geq 1$  then  $|\tilde{\eta}_2(A_1^{-(i+1)}x)| \leq \zeta(|A_1^{-(i+1)}x|) \sup_{|y|=1} |\eta(y)| \leq \|\eta\|_{\Gamma} \leq \|\eta\|_{\Gamma}(|A_1^{-(i+1)}x|)^{1+\mu}$ .

By hypothesis (3) we have

$$\begin{aligned} |A_1^{-(i+1)}\eta_1(A_1^i(x))| &\leq \|A_1^{-1}\|^{i+1} \|\eta\|_{\Gamma} |A_1^i x|^{1+\nu} \\ &\leq \|\eta\|_{\Gamma} \|A_1^{-1}\| \left( \|A_1^{-1}\| \|A_1\|^{1+\nu} \right)^i |x|^{1+\nu} \end{aligned}$$

and

$$\begin{aligned} |A_2^i \eta_2(A_1^{-(i+1)}(x))| &\leq \|A_2\|^i \|\eta\|_{\Gamma} (\|A_1^{-1}\|^{i+1} |x|)^{1+\mu} \\ &\leq \|\eta\|_{\Gamma} \|A_1^{-1}\|^{1+\mu} \left( \|A_2\| \|A_1^{-1}\|^{1+\mu} \right)^i |x|^{1+\mu}, \end{aligned}$$

which prove that the series in (8.4) and (8.5) are uniformly convergent and therefore their sums define continuous functions. From these bounds it is also clear that  $\Delta$  so obtained indeed belongs to  $\Gamma$  and  $\|\Delta\|_{\Gamma} \leq C\|\eta\|_{\Gamma}$ .  $\square$

We consider the operator  $\mathcal{N}: C_0^{1+\varepsilon} \times \Gamma \longrightarrow \Gamma$  defined by

$$\mathcal{N}(N, K^>) = N \circ ((\text{Id}, 0) + K^>).$$

We recall that  $C_0^{1+\varepsilon}$  is the subspace of  $C^{1+\varepsilon}$  consisting of the functions vanishing at the origin together with their derivative. Even though there are several available results on the differentiability of the composition operator, for the given topology of  $\Gamma$  we need to provide a proof.

**Proposition 8.5.** *Under the above conditions, we have that  $\mathcal{N}$  is  $C^1$  and*

$$D_2\mathcal{N}(N, K^>)\Delta = DN \circ ((\text{Id}, 0) + K^>) \Delta.$$

**Proof.** Let  $\alpha = \min(v, \mu)$  and  $\beta = \max(v, \mu)$ . Note that  $(1 + \alpha)(1 + \varepsilon) \geq 1 + \alpha + \varepsilon \geq 1 + \varepsilon \geq 1 + \beta$ . To shorten the notation we will write  $K = (\text{Id}, 0) + K^>$ .  $\mathcal{N}$  is linear with respect to  $N$ . We claim that  $N \mapsto \mathcal{N}(N, K^>)$  is continuous. Indeed, since

$$|N(K(x))| \leq \int_0^1 |DN(sK(x))K(x)| ds \leq \frac{\|N\|_{C^{1+\varepsilon}}}{1 + \varepsilon} \left(|x| + \|K^>\|_{\Gamma}|x|^{1+\alpha}\right)^{1+\varepsilon}$$

we see that  $\|\mathcal{T}(N, K^>)\|_{\Gamma} \leq C\|N\|_{C^{1+\varepsilon}}$ .

Now we prove that  $\mathcal{T}$  is  $C^1$  with respect to  $K^>$ . We first show that it is differentiable. This follows easily from the bound:

$$\begin{aligned} & |N(K(x) + \Delta(x)) - N(K(x)) - DN(K(x))\Delta(x)| \\ &= \left| \int_0^1 [DN(K(x) + s\Delta(x)) - DN(K(x))] \Delta(x) ds \right| \\ &\leq \int_0^1 \|N\|_{C^{1+\varepsilon}} s^{\varepsilon} |\Delta(x)|^{1+\varepsilon} ds \leq \frac{1}{1 + \varepsilon} \|N\|_{C^{1+\varepsilon}} \|\Delta\|_{\Gamma}^{1+\varepsilon} |x|^{(1+\alpha)(1+\varepsilon)}. \end{aligned}$$

Finally we show that  $D_2\mathcal{N}$  is continuous. We have to bound

$$\|D_2\mathcal{N}(N, K^>) - D_2\mathcal{N}(\bar{N}, \bar{K}^>)\|_{L(\Gamma, \Gamma)} = \sup_{\|\Delta\|_{\Gamma} \leq 1} \|[DN \circ K - D\bar{N} \circ \bar{K}]\Delta\|_{\Gamma}.$$

The continuity follows from

$$\begin{aligned} & [\|DN(K(x)) - D\bar{N}(K(x))\| + \|D\bar{N}(K(x)) - D\bar{N}(\bar{K}(x))\|] |\Delta(x)| \\ &\leq [\|DN - D\bar{N}\|_{C^{\varepsilon}} |K(x)|^{\varepsilon} + \|D\bar{N}\|_{C^{\varepsilon}} |K^>(x) - \bar{K}^>(x)|^{\varepsilon}] \|\Delta\|_{\Gamma} |x|^{1+\alpha} \\ &\leq \left[ \|N - \bar{N}\|_{C^{1+\varepsilon}} \left(|x| + \|K^>\|_{\Gamma}|x|^{1+\alpha}\right)^{\varepsilon} \right] \end{aligned}$$

$$\begin{aligned}
& + \|\bar{N}\|_{C^{1+\varepsilon}} \|K^> - \bar{K}^>\|_{\Gamma}^{\varepsilon} |x|^{(1+\alpha)\varepsilon} \Big] \|\Delta\|_{\Gamma} |x|^{1+\alpha} \\
& \leq \left[ (1 + \|K^>\|_{\Gamma} |x|^{\alpha})^{\varepsilon} \|N - \bar{N}\|_{C^{1+\varepsilon}} + \|\bar{N}\|_{C^{1+\varepsilon}} \|K^> - \bar{K}^>\|_{\Gamma}^{\varepsilon} \right] \\
& \quad \times \|\Delta\|_{\Gamma} |x|^{1+\alpha+\varepsilon}. \quad \square
\end{aligned}$$

Now we can easily finish the proof of Theorem 8.1. We define  $\mathcal{T}: C_0^{1+\varepsilon} \times \Gamma \rightarrow \Gamma$  by

$$\mathcal{T}(N, K^>) = K^> + S^{-1}N \circ ((\text{Id}, 0) + K^>).$$

We have that  $\mathcal{T}(0, 0) = 0$ . By Lemma 8.3 and Proposition 8.5 the operator  $\mathcal{T}$  is  $C^1$  and  $D_2\mathcal{T}(0, 0) = \text{Id}$ . Then we can apply the implicit function theorem to  $\mathcal{T}(N, K^>) = 0$  and obtain a neighborhood  $V$  of 0 in  $C_0^{1+\varepsilon}$  and a  $C^1$  function

$$K^{>,*} = (K_1^{>,*}, K_2^{>,*}) : V \subset C_0^{1+\varepsilon} \rightarrow \Gamma$$

such that  $\mathcal{T}(N, K^{>,*}(N)) = 0$  for all  $N \in V$ .

We note that the fact that  $|K_1^{>,*}(x)| \leq C|x|^{1+\nu}$  and  $|K_2^{>,*}(x)| \leq C|x|^{1+\mu}$  implies that  $K$  is differentiable at 0,  $DK_1^{>,*}(0) = 0$ ,  $DK_2^{>,*}(0) = 0$ . Therefore, the invariant manifold obtained is tangent to  $X_1$  at 0, simply because  $K = (\text{Id} + K_1^{>,*}, K_2^{>,*})$ .  $\square$

## 9. Non-resonant invariant manifolds for maps

The goal of this section is to study some non-resonant invariant manifolds of maps. The non-resonant invariant manifolds were introduced in [dIL97], and they include as particular cases the stable or strong stable manifolds. For optimal results concerning differentiability and also in the setting of Banach spaces we refer to [CFdIL03a, CFdIL03b]. An exposition of results that can be obtained using the graph transform is in [dIL03].

The concrete result that we prove in this section is Theorem 3.1. We will present first a proof in the analytic case, which is simpler, and then a proof in the finitely differentiable case.

**Remark 9.1.** For the readers familiar with linearization theorems, it may be useful to think of the non-resonant manifolds as a compromise between a Sternberg-type linearization theorem and the usual stable and unstable manifold theorems. If the map was indeed smoothly linearizable, any space invariant under the linearization would correspond to an invariant manifold. Unfortunately, requiring linearization is very restrictive both in terms of the non-resonance conditions required and in terms of the regularity (recall that, in general the smoothness of the linearization is only a fraction of the smoothness of the map).

There are several compromises possible. Let us discuss some of the alternatives and compare them with the present approach:

In [dlL97], the method used was to develop a partial normal form that, even if not full linearization, still possesses an invariant manifold. Then, the invariant manifold for the true map was obtained by using a graph transform. This gives essentially the same differentiability results as the approach given here. As pointed out in [dlL97], given the partial normal form, an alternative to the graph transform is to apply a Sternberg–Chen-type theorem [Ste57,BdlLW96] to obtain a smooth conjugation between the partial normal form and the map. The manifolds invariant under the partial normal form, have analogues invariant under the map. Nevertheless, since the regularity of the maps produced in the Sternberg–Chen theorem is only a fraction of the regularity of the original map, one can show much less regularity than the one obtained by the methods presented here. Also, since the linearizations are not unique, one cannot prove uniqueness by this method.

Besides the methods of normal forms, we would also like to mention the work [EIB98], which is based on reducing the problem to a fixed point problem in a carefully chosen Banach space.

We also note that, from the point of view of computer implementations, the computations of normal forms in the whole space require to manipulate functions with variables in the whole space. On the other hand, the parameterization method requires only to work with a number of variables given by the dimension of the manifold. This is particularly serious in applications to PDE's where the ambient dimension could be infinite but the non-resonant manifold could be of a small finite dimension.

### 9.1. Overview of the proof

We look for  $K$  of the form

$$K = K^{\leq} + K^{>}$$

where  $K^{\leq}(x) = \sum_{i=1}^L K_i x^{\otimes i}$  is a polynomial of degree  $L$ . We recall that  $K_i$  is a symmetric  $i$ -linear operator in  $E^{\otimes i}$  taking values in  $\mathbb{R}^d$ . Similarly, we will write  $R(x) = \sum_{i=1}^L R_i x^{\otimes i}$  where  $R_i$  is a symmetric  $i$ -linear operator in  $E^{\otimes i}$  taking values in  $E$ .

In Section 9.2, we will show that, under appropriate non-resonance conditions, it is possible to find  $K^{\leq}$  and  $R$  just matching powers of  $x$ . Then, the search for a  $K^{>}$  that leads to an invariant parameterization will be reduced to solving a non-linear equation in a Banach space, which will be discussed in Section 9.3 for the analytic case, and in Section 9.4 for the finite-differentiable case.

### 9.2. Solution of the formal problem

**Lemma 9.2.** Assume that  $(\text{Spec}(A_E))^i \cap \text{Spec}(A_C) = \emptyset$  for  $i = 2, \dots, L$  and that  $r \geq L$ . Then, we can find polynomials  $K^{\leq}$ ,  $R$  as before in such a way that

$$D^j(F \circ K^{\leq} - K^{\leq} \circ R)(0) = 0, \quad j = 0, \dots, L, \quad (9.1)$$



$$K^{\leq}(0) = 0, \quad DK^{\leq}(0) = (\text{Id}, 0), \quad (9.2)$$

$$R(0) = 0, \quad DR(0) = A_E. \quad (9.3)$$

Moreover, if we assume that  $N = F - A$  is sufficiently small, then  $K^{\leq} - (\text{Id}, 0)$  and  $R - A_E$  will be arbitrarily small.

This lemma is a simplified version of the normal form calculations which often appear in dynamical systems. A good reference for related results is [Nel69]. Of fundamental importance in the proof of the lemma are the operators (sometimes called Sylvester operators) defined on the space  $\mathbf{S}_i = \mathbf{S}_i(X, Y)$  of symmetric  $i$ -multilinear operators from  $X$  to  $Y$  ( $X, Y$  being vector spaces) by

$$\mathcal{L}_{A,B}^i M = AM - MB^{\otimes i}, \quad (9.4)$$

where  $A, B$  are linear maps.

A key result in the study of these operators is the following.

**Proposition 9.3.**

$$\begin{aligned} \text{Spec}(\mathcal{L}_{A,B}^i) &= \text{Spec}(A) - (\text{Spec}(B))^i \\ &= \{\lambda - \mu_1 \cdot \dots \cdot \mu_i \mid \lambda \in \text{Spec}(A), \mu_1, \dots, \mu_i \in \text{Spec}(B)\}. \end{aligned} \quad (9.5)$$

A proof of this proposition can be found in [Nel69]. In [CFdlL03a] (and also in [BK98]), one can find another proof and an analogue for Banach spaces. We note that in the generality of Banach spaces one has the inclusion  $\subset$  instead equality in (9.5).

**Proof of Proposition 9.3.** Note that since the left- and the right-hand sides of (9.5) are continuous with respect to the matrices  $A$  and  $B$ , it suffices to prove (9.5) for a dense set of matrices  $A$  and  $B$ . Hence, it suffices to establish the result when  $A$  and  $B$  are diagonalizable over the complex.

In such a case, we see that if  $(\lambda_j, e_j)$ ,  $(\mu_j, v_j)$  are eigenvalues and eigenvectors for  $A, B$ , respectively, given a set of indices  $\sigma_1, \dots, \sigma_i$ , and  $j$  we can consider the multilinear operator defined by

$$\Gamma_{\sigma_1, \dots, \sigma_i}^j(v_{\alpha_1} \otimes \dots \otimes v_{\alpha_i}) = \begin{cases} e_j & \text{if the sets } \{\alpha_1, \dots, \alpha_i\}, \{\sigma_1, \dots, \sigma_i\} \text{ are equal,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,  $\Gamma_{\sigma_1, \dots, \sigma_i}^j$  is symmetric and we have

$$\mathcal{L}_{A,B}^i \Gamma_{\sigma_1, \dots, \sigma_i}^j = (\lambda_j - \mu_{\sigma_1} \cdot \dots \cdot \mu_{\sigma_i}) \Gamma_{\sigma_1, \dots, \sigma_i}^j.$$

Hence,  $\Gamma_{\sigma_1, \dots, \sigma_i}^j$  is an eigenvector of  $\mathcal{L}_{A,B}^i$  of eigenvalue  $\lambda_j - \mu_{\sigma_1} \cdot \dots \cdot \mu_{\sigma_i}$ . Therefore we have the inclusion  $\text{Spec}(\mathcal{L}_{A,B}^i) \supset \text{Spec}(A) - (\text{Spec}(B))^i$ .

To prove the opposite inclusion, we note that the  $\Gamma$ 's obtained for different  $(\sigma, j)$ 's are linearly independent. Hence, since their number equals the dimension of the space of symmetric  $i$ -linear operators, we see that they are a complete set of eigenvectors. Hence, we have found all the spectrum of  $\mathcal{L}_{A,B}^i$ .  $\square$

**Remark 9.4.** Note that, as a particular case of Proposition 9.3, we obtain that, if  $i$  is such that  $(\max |\mu_j|)^i < \min |\lambda_j|$  (which happens for all

$$i > \frac{\log \min |\lambda_j|}{\log \max |\mu_j|},$$

when  $\max |\mu_j| < 1$ ), then  $\mathcal{L}_{A,B}^i$  is invertible.

**Proof of Lemma 9.2.** The case  $j = 0$  of the conclusions is satisfied when  $K^{\leq}(0) = 0$ ,  $R(0) = 0$ . Hence, we pick  $K_0 = 0$ ,  $R_0 = 0$ . The case  $j = 1$  amounts to  $AK_1 = K_1R_1$ . Hence, we take  $R_1 = A_E$  and  $K_1 = I_E$ , where  $I_E = (\text{Id}, 0)$  is the immersion of  $E$  into  $\mathbb{R}^d$ . In this way, we ensure (9.2) and (9.3).

The previous choice of  $R_1$  and  $K_1$  will only affect the smallness conditions that we have to impose on the non-linear terms, which as in previous sections, can be adjusted by scaling. This is, of course, analogous to our choice of a multiple of the eigenvector for the one-dimensional invariant manifolds of previous sections.

For  $j \geq 2$ , equating terms of order  $j$  in  $F \circ K - K \circ R = 0$  we obtain

$$AK_j - K_j A_E^{\otimes j} - K_1 R_j + P_j(K_1, \dots, K_{j-1}, R_1, \dots, R_{j-1}) = 0, \quad (9.6)$$

where  $P_j$  is a polynomial expression in its arguments. Taking projections over the spaces  $E$  and  $C$ , denoting by  $K_j^E = \Pi_E K_j$ , etc., and using the block notation for  $A$  as in (3.1), formula (9.6) becomes

$$\begin{aligned} A_E K_j^E + B K_j^C - K_j^E A_E^{\otimes j} - R_j + P_j^E &= 0, \\ A_C K_j^C - K_j^C A_E^{\otimes j} + P_j^C &= 0, \end{aligned} \quad (9.7)$$

where we have used that  $\Pi_C K_1 = 0$ .

Using the operators  $\mathcal{L}$  we can write (9.7) as

$$\mathcal{L}_{A_E, A_E}^j K_j^E = R_j - B K_j^C - P_j^E, \quad (9.8)$$

$$\mathcal{L}_{A_C, A_E}^j K_j^C = -P_j^C. \quad (9.9)$$

Note that, by Proposition 9.3, the hypotheses of Lemma 9.2 imply that  $\mathcal{L}_{A_C, A_E}^j$  is invertible. Now we follow the next iterative algorithm to solve (9.7). Assuming we already know  $K_i, R_i$  for  $1 \leq i < j$ ,

- (1) Since at this stage, the right-hand side of (9.9) is known and  $\mathcal{L}_{A_C, A_E}^j$  is invertible, we can obtain one and only one solution  $K_j^C$  of it.
- (2) We choose  $R_j$  in such a way that the right-hand side of (9.8) is in the range of  $\mathcal{L}_{A_E, A_E}^j$ .
- (3) We solve Eq. (9.8) for  $K_j^E$ .

A particular way to do (2) and (3) above, is to choose  $R_j := BK_j^C + P_j^E$  and then  $K_j^E = 0$ . Of course other procedures are possible. We can add to  $R_j$  terms in the range of  $\mathcal{L}_{A_E, A_E}^j$ , and we can add to  $K_j^E$  terms in the kernel of  $\mathcal{L}_{A_E, A_E}^j$ .

The last statement of the lemma just follows from observing that the polynomials  $P_j^E$  and  $P_j^C$  vanish when  $N \equiv 0$  and are continuous in the Taylor coefficients of  $N$ . This finishes the proof of Lemma 9.2.  $\square$

**Remark 9.5.** Hypotheses (2), (3) and (5) of Theorem 3.1 imply that if  $j > L$  the operator  $\mathcal{L}_{A_E, A_E}^j$  is invertible. In particular, when  $j > L$  we can take  $R_j = 0$ .

### 9.3. Proof of Theorem 3.1 when $r = \omega$

We discuss first the case  $r = \omega$ , that is when all the functions considered are analytic.

We take a norm in  $\mathbb{R}^d$  such that the associated operator norms  $\|A_E\|$  and  $\|A^{-1}\|$  verify  $\|A_E\| < \rho(A_E) + \varepsilon$  and  $\|A^{-1}\| < \rho(A^{-1}) + \varepsilon$  for some  $\varepsilon$  small enough, where  $\rho$  stands for the spectral radius (see in [CFdlL03a, Proposition A1] for details). Then, by hypothesis (5),

$$\|A^{-1}\| \|A_E\|^{L+1} < 1. \quad (9.10)$$

Note that the equation  $F \circ K = K \circ R$  can be written as

$$\mathcal{T}(N, K^>) := AK^{\leq} + AK^> + N \circ (K^{\leq} + K^>) - K^{\leq} \circ R - K^> \circ R = 0, \quad (9.11)$$

where we need to consider  $K^{\leq}$  and  $R$  as functionals of  $N$  which are computed precisely through the algorithm that we have indicated in the proof of Lemma 9.2. To prove Theorem 3.1 it suffices to consider small  $N$ , since we can always reduce to this case by scaling. This scaling technique will be considered in detail later.

Let

$$H_\delta^k = \left\{ G : \bar{B}_\delta(0) \subset E \rightarrow \mathbb{R}^d \mid G = \sum_{i=k}^{\infty} G_i x^{\otimes i}, \sum_{i=k}^{\infty} |G_i| \delta^i < \infty \right\}$$

endowed with the norm  $\|G\| := \sum_{i=k}^{\infty} |G_i| \delta^i$ . We consider  $\mathcal{T} : H_3^2 \times H_2^{L+1} \rightarrow H_2^{L+1}$  (here the space  $H_3^2$  corresponds to maps  $N$  from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ , instead of maps from  $E$  to  $\mathbb{R}^d$ ). Note that if  $N = 0$  then  $K^{\leq} = I_E$  and  $R = A_E$ , and hence  $\mathcal{T}(0, 0) = 0$ .

**Proposition 9.6.** *We have:*

- (1) *The operator  $\mathcal{T} : V \subset H_3^2 \times H_2^{L+1} \rightarrow H_2^{L+1}$  is analytic in a neighborhood  $V$  of  $(0, 0)$ .*
- (2)  $D_2\mathcal{T}(0, 0)\Delta = A\Delta - \Delta \circ A_E$ .

**Proof.** The fact that  $K^{\leq}, R$  are analytic in  $N$  is a consequence of the fact that they are algebraic expressions in a finite number of coefficients of  $N$ .

Also from Lemma 9.2, if  $N$  is small enough,  $R$  is a contraction. Therefore the operator  $\mathcal{T}$  is well defined.

Now, we use that on the set  $H_3^k \times \{G \in H_\delta^\ell \mid \|G\| < 2\}$ , the map  $(H, G) \mapsto H \circ G$  is analytic (see [Mey75] for more details). The idea is that if we interpret the series  $H \circ G = \sum_i H_i G^i$  as a series in the Banach algebra of analytic functions, we can bound the norm of the general term by  $\|H_i G^i\| \leq \|H_i\| \|G\|^i$ , and hence the series obtained summing  $H_i G^i$  converges as a series of elements in the Banach algebra.

Applying this fact to  $H = N, G = K^{\leq} + K^>$ , we obtain that the third term in (9.11) is analytic. Applying it when  $H = K^>, G = R$  we obtain the analyticity of the last term in (9.11).  $\square$

**Lemma 9.7.** *Under hypothesis (5) of Theorem 3.1, we have that  $D_2\mathcal{T}(0, 0)$  from  $H_2^{L+1}$  to  $H_2^{L+1}$  is boundedly invertible.*

**Proof.** If we consider the equation

$$A\Delta - \Delta \circ A_E = \eta \quad (9.12)$$

with  $\eta = \sum_{i=L+1}^{\infty} \eta_i x^{\otimes i}$ , we see that

$$\Delta_i - A^{-1} \Delta_i A_E^{\otimes i} = A^{-1} \eta_i, \quad i \geq L+1. \quad (9.13)$$

By hypothesis (2) and (9.10) we have that  $\|A^{-1}\| \|A_E\|^i < 1$ , for  $i \geq L+1$ . Therefore (9.13) defines  $\Delta$  uniquely and we have

$$|\Delta_i| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|A_E\|^i} |\eta_i| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|A_E\|^{L+1}} |\eta_i|,$$

and hence

$$\sum_{i \geq L+1}^{\infty} |\Delta_i| 2^i \leq C \sum_{i \geq L+1}^{\infty} |\eta_i| 2^i. \quad \square$$

We can now finish the proof of Theorem 3.1 when  $r = \omega$ . Proposition 9.6 and Lemma 9.7 give that the hypotheses of the implicit function theorem for  $\mathcal{T}(N, K^>) = 0$  near  $(0, 0)$  are satisfied, and therefore we have established Theorem 3.1 for  $N$  small enough.

Theorem 3.1 for a general  $N$  can be obtained by observing that if we consider  $F^\delta = (1/\delta)F(\delta x)$  for  $\delta$  small enough, then  $N^\delta$  is small in the sense needed by the result hitherto proved. Hence, we obtain  $K$  and  $R$  analytic satisfying

$$F^\delta \circ K = K \circ R.$$

It is immediate to verify that  $K^{1/\delta}(x) = \delta K(\frac{1}{\delta}x)$ ,  $R^{1/\delta}(x) = \delta R(\frac{1}{\delta}x)$  verify

$$F \circ K^{1/\delta} = K^{1/\delta} \circ R^{1/\delta}$$

in a neighborhood of 0, as well as the other claims.

#### 9.4. Proof of Theorem 3.1 when $r \in \mathbb{N}$

The method used in the previous section does not directly work in this case because the term  $(K^>, R) \mapsto K^> \circ R$  ceases to be differentiable with respect to  $R$  when we give  $K^>$  and  $K^> \circ R$  the  $C^r$  topology with  $r \in \mathbb{N}$ . Simply note that when differentiating  $K^> \circ R$  formally with respect to  $R$ , the term  $DK^>$  appears and does not belong to  $C^r$ , but just to  $C^{r-1}$ .

We write  $N = N^{\leq} + N^>$ , where  $N^{\leq}$  is the Taylor polynomial of  $N$  up to order  $L$ , and we denote by  $K^0, R^0$  the solution of  $(A + N^{\leq}) \circ K^0 = K^0 \circ R^0$  which is obtained applying the analytic result—already established—to  $F^{\leq} = A + N^{\leq}$  in place of  $F = A + N$ . Then we look for  $K$  in the form  $K = K^0 + K^>$ .

Let  $\Sigma$  be the space of polynomials  $Q: \mathbb{R}^d \rightarrow \mathbb{R}^d$  of degree less than or equal to  $L$  such that  $Q(0) = 0$  and  $DQ(0) = 0$  and, for  $r \geq L + 1$ , let

$$C_L^r(\bar{B}_\rho(0)) = \{f \in C^r(\bar{B}_\rho(0)) \mid D^i f(0) = 0, 0 \leq i \leq L\}$$

endowed with the topology given by

$$|||f||| := \max_{L+1 \leq i \leq r} \sup_{|x| \leq \rho} \|D^i f(x)\|.$$

It is clearly a Banach space. We consider the operator

$$\mathcal{N}: V \subset \Sigma \times C_L^{r+1}(\bar{B}_2(0)) \times C_L^r(\bar{B}_1(0)) \longrightarrow C_L^r(\bar{B}_1(0))$$

in a neighborhood  $V$  of  $(0, 0, 0)$  defined by

$$\begin{aligned}\mathcal{N}(N^{\leq}, N^>, K^>) &= AK^0 + AK^> + N^{\leq} \circ (K^0 + K^>) \\ &\quad + N^> \circ (K^0 + K^>) - K^0 \circ R^0 - K^> \circ R^0,\end{aligned}\quad (9.14)$$

where  $K^0$  and  $R^0$  are considered as analytic functions of  $N^{\leq}$ .

With these notations, the original problem becomes  $\mathcal{N}(N^{\leq}, N^>, K^>) = 0$ . Since  $\mathcal{N}(0, 0, 0) = 0$ , again we try to construct the solution by using the implicit function theorem.

**Proposition 9.8.** *The operator  $\mathcal{N}$  defined in (9.14) is continuous with respect to all three variables and  $C^1$  with respect to  $K^>$ . Moreover,*

$$D_3\mathcal{N}(0, 0, 0)\Delta = A\Delta - \Delta \circ A_E. \quad (9.15)$$

**Proof.** We note that the linear operator  $H_2^1 \rightarrow C_0^{r+1}(\bar{B}_1(0))$  sending a map to itself is continuous and thus differentiable. Then we can consider  $\mathcal{N}$  as being the composition of the maps

$$V \subset \Sigma \times C_L^{r+1}(\bar{B}_1(0)) \times C_L^r(\bar{B}_1(0)) \rightarrow \Sigma \times H_2^1 \times H_2^1 \times C_L^{r+1}(\bar{B}_1(0)) \times C_L^r(\bar{B}_1(0))$$

sending  $(N^{\leq}, N^>, K^>)$  to  $(N^{\leq}, K^0, R^0, N^>, K^>)$ , where  $K^0$  and  $R^0$  are the maps obtained applying the analytic result to  $F^{\leq} = A + N^{\leq}$ , which depend analytically on  $N^{\leq}$ , the injection

$$\begin{aligned}\Sigma \times H_2^1 \times H_2^1 \times C_L^{r+1}(\bar{B}_1(0)) \times C_L^r(\bar{B}_1(0)) \\ \longrightarrow \Sigma \times C_0^{r+1}(\bar{B}_1(0)) \times C_0^{r+1}(\bar{B}_1(0)) \times C_L^{r+1}(\bar{B}_1(0)) \times C_L^r(\bar{B}_1(0))\end{aligned}$$

and finally the map

$$\begin{aligned}\tilde{V} \subset \Sigma \times C_0^{r+1}(\bar{B}_1(0)) \times C_0^{r+1}(\bar{B}_1(0)) \times C_L^{r+1}(\bar{B}_1(0)) \times C_L^r(\bar{B}_1(0)) \\ \rightarrow C^r(\bar{B}_1(0))\end{aligned}\quad (9.16)$$

sending  $(N^{\leq}, K^0, R^0, N^>, K^>)$  to  $\mathcal{N}(N^{\leq}, N^>, K^>)$ . We restrict  $V$  and  $\tilde{V}$  in order to have the maps well defined. Since the space  $C_L^r$  is a closed linear subspace of  $C^r$  and the norm there is the restriction of the  $C^r$  norm, we obtain that the differentiability results in [dlLO99] also hold for  $C_L^r$ . Then the map (9.16) is continuous (we are working with sets of maps defined in spaces of finite dimension) and it is  $C^1$  with respect to  $K^>$ . Hence  $\mathcal{N}$  is continuous with respect to its three variables and  $C^1$  with respect to  $K^>$ . Moreover, by the definitions of  $K^0$  and  $R^0$  the range of  $\mathcal{N}$  is contained

in  $C_L^r(\bar{B}_1(0))$ . Actually we have that

$$\begin{aligned} D_3\mathcal{N}(N^{\leq}, N^>, K^>)\Delta \\ = [A + DN^{\leq} \circ (K^0 + K^>) + DN^> \circ (K^0 + K^>)]\Delta - \Delta \circ R^0. \quad \square \end{aligned}$$

The following is slightly weaker than Lemma 5 of [BdlW96]. The proof, however, is simpler.

**Lemma 9.9.** *The operator  $D_3\mathcal{N}(0, 0, 0)$  is boundedly invertible as an operator from  $C_L^r(\bar{B}_1(0))$  to itself.*

**Proof.** We have to solve  $A\Delta - \Delta \circ A_E = \eta$  which is equivalent to

$$\Delta - A^{-1}\Delta \circ A_E = A^{-1}\eta.$$

We introduce the operator  $\mathcal{A}$  defined by  $\mathcal{A}\Delta = A^{-1}\Delta \circ A_E$ . We claim that the norm of  $\mathcal{A}$  considered as an operator from  $C_L^r(\bar{B}_1(0))$  to itself is strictly smaller than 1. Indeed, if  $L+1 \leq i \leq r$

$$\|D^i(A^{-1}\Delta \circ A_E)\|_{C^0} \leq \|A^{-1}\| \|D^i\Delta\|_{C^0} \|A_E\|^i \leq \|A^{-1}\| \|D^i\Delta\|_{C^0} \|A_E\|^{L+1} \quad (9.17)$$

and hence  $\|\mathcal{A}\Delta\| \leq \|A^{-1}\| \|A_E\|^{L+1} \|\Delta\|$ . Therefore, we have that  $\Delta = (\text{Id} - \mathcal{A})^{-1}(A^{-1}\eta)$ .  $\square$

Proposition 9.8 and Lemma 9.9 establish the hypotheses of the generalized version of the implicit function theorem, which assumes continuity of the map and being  $C^1$  with respect to the variable that one wants to isolate, but only provides continuity of the implicit function that it defines (see [Nir01]). Applying this to  $\mathcal{N}(N^{\leq}, N^>, K^>) = 0$  near  $(0, 0, 0)$  we get a continuous map  $K^> = \mathcal{M}(N^{\leq}, N^>)$  defined in a neighborhood of  $(0, 0)$ .

Now, given a map  $F$  satisfying the hypotheses of Theorem 3.1, we scale it to  $F^\delta = A + N^{\leq, \delta} + N^>, \delta$  with  $\delta$  so small that  $(N^{\leq, \delta}, N^>, \delta)$  belongs to the domain of  $\mathcal{M}$ . The parameterization  $K = K^0 + K^>$  thus obtained is the solution of  $F^\delta \circ K = K \circ R^0$ , where  $K^0$  and  $R^0$  are the analytic maps depending on  $F^{\leq, \delta} = A + N^{\leq, \delta}$  provided by the proof in the analytic case.

## 10. Non-resonant invariant manifolds for differential equations

The results we have proved in the previous section translate to results for flows using the argument mentioned at the end of Section 2. Nevertheless, it is interesting, specially from the point of view of implementing algorithms, to give direct proofs of the results for differential equations. We will see that the leading ideas and methods are very similar to those for maps.

The result we deal within this section is Theorem 3.2, which we prove following the parameterization method.

For differential equations  $x' = \mathcal{X}(x)$  such that  $\mathcal{X}(0) = 0$ , if we have an invariant subspace  $E$  by  $D\mathcal{X}(0)$ , we look for a parameterization  $K$  and a polynomial  $R$  defined in  $E$  such that

$$\mathcal{X} \circ K = DK \cdot R, \quad (10.1)$$

that is, we ask the vector field  $\mathcal{X}$  on the image of  $K$  to be the pull forward of a vector field  $R$  in  $E$ .

Following the same strategy as for the case of maps we look for  $K$  of the form

$$K = K^{\leq} + K^{>},$$

where  $K^{\leq}$  is a polynomial of degree  $L$  and  $K^{>}$  is a function vanishing at the origin together with its first  $L$  derivatives.

The polynomials  $K^{\leq}$  and  $R$  will be found matching powers in (10.1). Then, we will write a functional equation for  $K^{>}$  whose solution will be found by applying the implicit function theorem in an appropriate Banach space.

### 10.1. Formal solution

We summarize the formal calculations needed to find  $K^{\leq}$  and  $R$  in the following result.

**Lemma 10.1.** *Given  $\mathcal{X}: U \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\mathcal{X}(0) = 0$ ,  $\mathcal{X} \in C^L$ , satisfying hypothesis (3) of Theorem 3.2, we can find polynomials  $K^{\leq}$  and  $R$  of degree not bigger than  $L$  such that*

$$D^j(\mathcal{X} \circ K^{\leq} - DK^{\leq} \cdot R)(0) = 0, \quad 0 \leq j \leq L, \quad (10.2)$$

$$K^{\leq}(0) = 0, \quad DK^{\leq}(0)E = E, \quad (10.3)$$

$$R(0) = 0, \quad DR(0) = A_E. \quad (10.4)$$

Moreover, if we assume that  $N = F - A$  and  $B$  are sufficiently small, then  $K^{\leq} - (\text{Id}, 0)$  and  $R - A_E$  are arbitrarily small.

For the calculations in Lemma 10.1 we will use the operators  $\tilde{\mathcal{L}}_{A,B}^i$  from the space  $\mathbf{S}_i$  of symmetric  $i$ -linear operators in  $E$  with values in  $E$  (or  $\mathbb{R}^d$ ), defined by

$$(\tilde{\mathcal{L}}_{A,B}^i K)(x) = AK(x) - DK(x)Bx, \quad (10.5)$$

for  $x \in E$ . These operators are similar to the ones used in Section 9, but their spectrum is different.



**Proposition 10.2.**

$$\begin{aligned} \text{Spec}(\tilde{\mathcal{L}}_{A,B}^i) &= \text{Spec}(A) - i \text{Spec}(B) \\ &:= \{\lambda - (\mu_1 + \mu_2 + \cdots + \mu_i) \mid \lambda \in \text{Spec}(A), \mu_1, \dots, \mu_i \in \text{Spec}(B)\}. \end{aligned} \quad (10.6)$$

**Proof.** The proof is completely analogous to the one of Proposition 9.3. Because of the continuity of the objects of (10.6) with respect to  $A$  and  $B$ , it is sufficient to prove (10.6) for the dense subset of diagonalizable matrices  $A$  and  $B$ .

Assuming that  $A$  and  $B$  are diagonalizable, let  $(\lambda_j, e_j)$  and  $(\mu_j, v_j)$  be the eigenvalues and eigenvectors of  $A$  and  $B$  respectively. Given indices  $\sigma_1, \dots, \sigma_i$  and  $\ell$ , let the  $i$ -linear symmetric form  $\Gamma_{\sigma_1, \dots, \sigma_i}^\ell \in \mathbf{S}_i$  be defined by

$$\Gamma_{\sigma_1, \dots, \sigma_i}^\ell(v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_i}) = \begin{cases} e_\ell & \text{if the sets } \{\alpha_1, \dots, \alpha_i\} \text{ and } \{\sigma_1, \dots, \sigma_i\} \text{ are equal,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the  $\Gamma$ 's so defined are linearly independent and form a basis of  $\mathbf{S}_i$ . An easy calculation gives that  $\tilde{\mathcal{L}}_{A,B}^i \Gamma_{\sigma_1, \dots, \sigma_i}^\ell = (\lambda_\ell - (\mu_{\sigma_1} + \cdots + \mu_{\sigma_i})) \Gamma_{\sigma_1, \dots, \sigma_i}^\ell$ .

This proves that  $\lambda_\ell - (\mu_{\sigma_1} + \cdots + \mu_{\sigma_i})$  is an eigenvalue of  $\tilde{\mathcal{L}}_{A,B}^i$ . Since the  $\Gamma$ 's so obtained form a basis of eigenvectors of  $\mathbf{S}_i$ , (10.6) holds true.  $\square$

**Remark 10.3.** Note that if the relations

$$\lambda - (\mu_1 + \cdots + \mu_i) \neq 0$$

hold for  $\lambda \in \text{Spec}(A)$ ,  $\mu_1, \dots, \mu_i \in \text{Spec}(B)$  then  $\tilde{\mathcal{L}}_{A,B}^i$  is invertible, since none of its eigenvalues is zero.

**Proof of Lemma 10.1.** We look for

$$K^{\leq}(x) = \sum_{j=0}^L K_j x^{\otimes j}, \quad R(x) = \sum_{j=0}^L R_j x^{\otimes j} \quad (10.7)$$

in such a way to satisfy  $\mathcal{X} \circ K = DK \cdot R$  up to order  $L$ . We denote by  $\Pi_E$  and  $\Pi_C$  the projections onto  $E$  and  $C$ , respectively, and  $K_j^E = \Pi_E K_j$ ,  $K_j^C = \Pi_C K_j$ .

Taking  $K_0 = 0$ ,  $K_1^E = \text{Id}$ ,  $K_1^C = 0$ ,  $R_0 = 0$ ,  $R_1 = A_E$  we have that (10.2) is satisfied for  $j = 0, 1$ .

After projecting Eq. (10.2) to  $E$  and  $C$ , we obtain the following relations for the terms of order  $j$ ,  $2 \leq j \leq L$ :

$$\begin{aligned} A_E K_j^E x^{\otimes j} + B K_j^C x^{\otimes j} &= K_1^E R_j x^{\otimes j} + D[K_j^E x^{\otimes j}] R_1 x + (P_j^E) x^{\otimes j}, \\ A_C K_j^C x^{\otimes j} &= K_1^C R_j x^{\otimes j} + D[K_j^C x^{\otimes j}] R_1 x + (P_j^C) x^{\otimes j}, \end{aligned} \quad (10.8)$$

where  $P_j^{E,C} = P_j^{E,C}(K_1, \dots, K_{j-1}, R_1, \dots, R_{j-1})$  are polynomial in their arguments, and their coefficients vanish if  $N$  and  $B$  vanish.

We rewrite (10.8) in the form

$$\tilde{\mathcal{L}}_{A_E, A_E}^j K_j^E = -BK_j^C + R_j + P_j^E, \quad (10.9)$$

$$\tilde{\mathcal{L}}_{A_C, A_E}^j K_j^C = P_j^C. \quad (10.10)$$

We can solve these equations inductively, beginning with  $j = 2$ . We will follow the procedure:

- (1) Solve Eq. (10.10). By hypothesis (3) of Theorem 3.2 and Proposition 10.2, this is indeed possible and moreover the  $K_j^C$  obtained is unique.
- (2) Choose  $R_j$  so that  $-BK_j^C + R_j + P_j^E$  belongs to the range of  $\tilde{\mathcal{L}}_{A_E, A_E}^j$ . Then it is possible to
- (3) Solve Eq. (10.9) to find  $K_j^E$ .

A particular way to do (2) and (3), is to choose  $R_j := BK_j^C - P_j^E$  and then  $K_j^E = 0$ . Of course other procedures are possible. We can add to  $R_j$  terms in the range of  $\tilde{\mathcal{L}}_{A_E, A_E}^j$ , and we can add to  $K_j^E$  terms in the kernel of  $\tilde{\mathcal{L}}_{A_E, A_E}^j$ .

The last claim in Lemma 10.1 follows because the polynomials  $P^E$  and  $P^C$  are zero when  $N$  and  $B$  are zero.  $\square$

## 10.2. The non-linear problem

Let  $\mu_+ = \sup\{\operatorname{Re} \mu \mid \mu \in \operatorname{Spec}(A_E)\} < 0$  and  $\lambda_- = \inf\{\operatorname{Re} \lambda \mid \lambda \in \operatorname{Spec}(A)\}$ . By the fact that  $\mu_+ < 0$  and hypothesis (4) of Theorem 3.2, there exists  $\varepsilon > 0$  small enough such that

$$\mu_+ + \varepsilon < 0, \quad (10.11)$$

$$-\lambda_- + (L+1)\mu_+ + (L+2)\varepsilon < 0. \quad (10.12)$$

We take a norm in  $\mathbb{C}^d$  such that

$$|e^{A_E t} y| \leq e^{(\mu_+ + \varepsilon/2)t} |y|, \quad t \geq 0, \quad (10.13)$$

for all  $y \in \mathbb{R}^d$ , and such that  $\|B\|$  is as small as we need (see [CFdl03a, Appendix A]).

We will work with a scaled vector field

$$\mathcal{X}^\delta(x) = (1/\delta)\mathcal{X}(\delta x) = Ax + N^\delta(x),$$

with  $\delta$  small enough such that  $N^\delta$  is sufficiently small to fulfill the smallness requirements that we will need, and also such that  $\mathcal{X}^\delta$  is defined in the ball  $\bar{B}_3(0) \subset \mathbb{C}^d$ .

Clearly if we find  $K$  and  $R$  satisfying  $\mathcal{X}^\delta \circ K = DK \cdot R$  then  $\bar{K}(x) = \delta K(x/\delta)$  and  $\bar{R}(x) = \delta R(x/\delta)$  will satisfy

$$\mathcal{X} \circ \bar{K} = D\bar{K} \cdot \bar{R}.$$

Introducing  $\mathcal{X} = A + N$  and  $K = K^\leq + K^>$  in  $\mathcal{X} \circ K - DK \cdot R = 0$ , we obtain

$$\mathcal{U}(N, K^>) := AK^\leq + AK^> + N \circ (K^\leq + K^>) - DK^\leq \cdot R - DK^> \cdot R = 0, \quad (10.14)$$

where  $K^\leq$  and  $R$  depend on  $N$  (in fact they only depend on the Taylor polynomial of degree  $L$  of  $N$  at the origin).

### 10.3. The analytic case

We will work with the spaces of analytic functions

$$H_\delta^k = \left\{ F : \bar{B}_\delta(0) \rightarrow \mathbb{C}^d \left| F(x) = \sum_{j=k}^{\infty} F_j x^{\otimes j}, \sum_{j=k}^{\infty} \|F_j\| \delta^j < \infty \right. \right\},$$

endowed with the norm  $\|F\| := \sum_{j=k}^{\infty} \|F_j\| \delta^j$ .

**Proposition 10.4.** *If  $N$  is analytic, then:*

- (a) *The operator  $\mathcal{U} : V \subset H_3^2 \times H_2^{L+1} \longrightarrow H_2^{L+1}$  is analytic in a neighborhood  $V$  of  $N = 0, K^> = 0$ .*
- (b)  $D_2\mathcal{U}(0, 0)\Delta = A\Delta - (D\Delta) \cdot A_E$ .

**Proof.** We have that  $K^\leq$  and  $R$  depend analytically on  $N$ . Since  $N$  is as small as we want,  $K^\leq$  and  $R$  will be as close as we want to the immersion of  $E$  in  $\mathbb{C}^d$  and  $A_E$ , respectively. Therefore the composition in (10.14) is well defined.

The results in [Mey75] assure that  $\mathcal{U}$  is analytic and give the formula for the derivative.  $\square$

**Remark 10.5.** Although  $\mathcal{U}$  is analytic, the fact that  $\mathcal{U}$  is  $C^1$  will be sufficient for our purposes.

**Lemma 10.6.** *The operator*

$$S\Delta := D_2\mathcal{U}(0, 0)\Delta = A\Delta - (D\Delta) \cdot A_E$$

*is boundedly invertible from  $H_2^{L+1}$  to itself.*

**Proof.** To prove that  $\mathcal{S}$  is invertible, given  $\eta \in H_2^{L+1}$  we have to find a unique  $\Delta \in H_2^{L+1}$  which solves the equation  $\mathcal{S}\Delta = \eta$ , that is,

$$D\Delta(x) \cdot A_E x - A\Delta(x) = -\eta(x). \quad (10.15)$$

We can deal with Eq. (10.15) using power series but we prefer the following method which provides an explicit formula for the solution  $\Delta$ . This formula has the advantage that it also makes sense in other differentiabilitys. In addition, we will see that it is remarkably similar to those of previous sections.

We introduce  $x = e^{A_E t} y$  and

$$\tilde{\Delta}(t, y) = \Delta(e^{A_E t} y). \quad (10.16)$$

By (10.13),  $\tilde{\Delta}$  is well defined for  $|y| \leq 2$  and  $t \geq 0$ .

Eq. (10.15) becomes

$$\frac{d}{dt} \tilde{\Delta}(t, y) - A \tilde{\Delta}(t, y) = -\eta(e^{A_E t} y),$$

which can be integrated

$$\tilde{\Delta}(t, y) = e^{At} \left[ e^{-At_0} \tilde{\Delta}(t_0, y) - \int_{t_0}^t e^{-As} \eta(e^{A_E s} y) ds \right], \quad t_0, t \in [0, \infty). \quad (10.17)$$

We have that there exists  $M > 1$  such that

$$|e^{-At} y| \leq M e^{-(\lambda_- - \varepsilon)t} |y|, \quad t \geq 0. \quad (10.18)$$

Therefore,

$$\begin{aligned} |e^{-At_0} \Delta(e^{A_E t_0} y)| &\leq |e^{-At_0}| \sum_{j=L+1}^{\infty} \|\Delta_j\| |e^{A_E t_0} y|^j \\ &\leq M e^{-(\lambda_- - \varepsilon)t_0} \sum_{j=L+1}^{\infty} \|\Delta_j\| e^{j(\mu_+ + \varepsilon/2)t_0} |y|^j \\ &= M \sum_{j=L+1}^{\infty} \|\Delta_j\| e^{[-\lambda_- + (L+1)\mu_+ + (L+2)\varepsilon]t_0} e^{(j-L-1)(\mu_+ + \varepsilon)t_0} |y|^j, \end{aligned}$$

which converges for  $t_0 \geq 0$ , and tends to zero as  $t_0$  tends to  $+\infty$ . Taking  $t = 0$  in (10.17) and letting  $t_0 \rightarrow +\infty$ , we deduce

$$\Delta(y) = \tilde{\Delta}(0, y) = \int_0^\infty e^{-As} \eta(e^{A_E s} y) ds. \quad (10.19)$$

Since  $\eta = \sum_{j=L+1}^\infty \eta_j x^{\otimes j} \in H_2^{L+1}$ , we have

$$\Delta(y) = \int_0^\infty e^{-As} \sum_{j=L+1}^\infty \eta_j (e^{A_E s} y)^{\otimes j} ds. \quad (10.20)$$

The fact that the improper integral in (10.20) converges uniformly can be checked with the same type of estimates as before using now (10.11)–(10.13), which ensure that

$$\sup_{j \geq L+1} \int_0^\infty \|e^{-As}\| \|e^{A_E s}\|^j ds =: C < \infty.$$

Finally we have that

$$\begin{aligned} |\Delta_j y^{\otimes j}| &= \left| \int_0^\infty e^{-As} \eta_j (e^{A_E s} y)^{\otimes j} ds \right| \leq \int_0^\infty \|e^{-As}\| \|\eta_j\| |e^{A_E s} y|^j ds \\ &\leq \|\eta_j\| |y|^j \int_0^\infty \|e^{-As}\| \|e^{A_E s}\|^j ds, \end{aligned}$$

and hence  $\|\Delta_j\| \leq C \|\eta_j\|$  and  $\|\Delta\|_{H_2^{L+1}} \leq C \|\eta\|_{H_2^{L+1}}$ .  $\square$

After the previous considerations, the proof of Theorem 3.2 in the analytic case follows from a straightforward application of the implicit function theorem.

#### 10.4. The finite-differentiable case

In contrast with the proof of Theorem 3.1, here a new difficulty arises. For a fixed non-linear part  $N$  of the vector field, the operator  $\mathcal{U}(N, \cdot)$  in (10.14) sends  $C^{r+1}$  maps to  $C^r$  maps and it is not at all clear how to choose Banach spaces in which  $D_2 \mathcal{U}$  is invertible.

To overcome these difficulties we do the following. We consider the auxiliary vector field  $\mathcal{X}^0(x) = Ax + N^\leq(x)$  where  $N^\leq(x)$  is the Taylor polynomial of  $N$  of degree  $L$ . Since it is a polynomial vector field, the previous work on the analytic case applies.

Then there exist an analytic  $K^0$  and a polynomial  $R^0$  such that

$$\mathcal{X}^0 \circ K^0 - DK^0 \cdot R^0 = 0.$$

Then we look for  $K = K^0 + K^>$ , with  $\sup_x |K^>(x)|/|x|^{L+1} < \infty$ , such that

$$\mathcal{X} \circ K - DK \cdot R^0 = 0. \quad (10.21)$$

To overcome the difficulty that we have mentioned above, we look for an equivalent integrated version of Eq. (10.21). Let  $\varphi(t, y)$  be the flow of

$$x' = R^0(x) = A_E x + R^N(x),$$

where  $R^N := R^0 - A_E$ . From the proof of the analytic case it is clear that if  $N$  is small then  $R^N$  is small. Also let

$$\psi(t) = e^{-At}.$$

First we establish some simple estimates on  $\varphi$ .

**Lemma 10.7.** *Let  $B_\rho(0) \subset E$ ,  $\rho > 0$ , and  $\varepsilon > 0$ . Under hypothesis (2), and smallness on  $N$  and  $\varepsilon$ , we have that there exist constants  $M_j$  such that*

$$|\varphi(t, x)| \leq e^{(\mu_+ + \varepsilon)t} |x|, \quad \forall x \in B_\rho(0), \quad \forall t \geq 0,$$

$$\|D_x^j \varphi(t, x)\| \leq M_j e^{(\mu_+ + \varepsilon)t}, \quad \forall x \in B_\rho(0), \quad \forall t \geq 0, \quad (10.22)$$

for  $1 \leq j \leq L$ . Furthermore,  $M_1 = 1$  and for  $2 \leq j \leq L$ ,  $M_j$  are small if  $N^\leq$  is small.

**Proof.** For  $j = 0$  the result is a well known estimate in Lyapunov stability theory. One uses that  $|e^{A_E t} x| \leq e^{(\mu_+ + \varepsilon/2)t} |x|$  and  $\|DR^N\|_{C^0} \leq \varepsilon/2$ .

For  $j = 1$ , from the variational equation

$$D_x \varphi(t, x)' = [A_E + DR^N(\varphi(t, x))] D_x \varphi(t, x), \quad D_x \varphi(0, x) = \text{Id},$$

and since  $DR^N$  has norm less than  $\varepsilon/2$ , we get

$$\|D_x \varphi(t, x)\| \leq e^{(\mu_+ + \varepsilon)t}.$$

Moreover, if  $\phi(t)$  is a fundamental matrix of

$$x' = [A_E + DR^N(\varphi(t, x))]x,$$

we have

$$\|\phi(t)\phi^{-1}(s)\| \leq e^{(\mu_+ + \varepsilon)(t-s)}, \quad t \geq s. \quad (10.23)$$

The higher order variational equations have the form

$$D_x^j \varphi(t, x)' = [A_E + DR^N(\varphi(t, x))]D_x^j \varphi(t, x) + P_j(t, x), \quad D_x^j \varphi(0, x) = 0, \quad (10.24)$$

with

$$P_j(t, x) = \sum_{i=2}^j \sum_{\substack{1 \leq \ell_1, \dots, \ell_i \leq j \\ \ell_1 + \dots + \ell_i = j}} c_{\ell_1, \dots, \ell_i}^{j,i} D^i R^N(\varphi(t, x)) D_x^{\ell_1} \varphi(t, x) \cdots D_x^{\ell_i} \varphi(t, x), \quad (10.25)$$

where  $c_{\ell_1, \dots, \ell_i}^{j,i}$  are combinatorial constants. Note that  $P_j(t, x)$  is a  $j$ -linear map. The solution of (10.24) is given by

$$D_x^j \varphi(t, x) = \int_0^t \phi(t)\phi^{-1}(s)P_j(s, x)ds. \quad (10.26)$$

From (10.26), (10.23) and (10.25) one easily checks inductively (10.22).  $\square$

To find an equation for  $K^>(x) = O(|x|^{L+1})$  equivalent to (10.21), we evaluate (10.21) at  $\varphi(t, x)$ , we multiply by  $\psi(t) = e^{-At}$  and we sum and subtract the term  $\psi'(t)K^>(\varphi(t, x))$ :

$$\begin{aligned} & \psi(t)\mathcal{X}(K(\varphi(t, x))) \\ & - \psi(t)DK^0(\varphi(t, x))R^0(\varphi(t, x)) + \psi'(t)K^>(\varphi(t, x)) \\ & - \psi(t)DK^>(\varphi(t, x))R^0(\varphi(t, x)) - \psi'(t)K^>(\varphi(t, x)) = 0. \end{aligned}$$

We use that  $DK^0 \cdot R^0 = \mathcal{X}^0 \circ K^0$  and that the fourth and fifth terms together are equal to  $-\frac{d}{dt}[\psi(t)K^>(\varphi(t, x))]$ . By

$$|\psi(t)K^>(\varphi(t, x))| \leq \|\psi(t)\| \|K^>\| |\varphi(t, x)|^{L+1} \leq C e^{(-\lambda_- + \varepsilon)t} e^{(L+1)(\mu_+ + \varepsilon)t}$$

and condition (10.12), it makes sense to integrate with respect to  $t$  from 0 to  $\infty$ . We deduce that if  $K$  satisfies (10.21), then  $K^>$  satisfies

$$K^>(x) + \int_0^\infty \psi(t) [\mathcal{X}(K^0 + K^>) - \mathcal{X}^0(K^0) - AK^>](\varphi(t, x))dt = 0. \quad (10.27)$$

Conversely, if  $K^>$  satisfies (10.27) then

$$\begin{aligned} DK^>(\varphi(t, x))R^0(\varphi(t, x)) \\ &= - \int_0^\infty \psi(s) \left[ D\mathcal{X}(K^0 + K^>)D(K^0 + K^>) - D\mathcal{X}^0(K^0)DK^0 - ADK^> \right] \\ &\quad \circ \varphi(s, \varphi(t, x))D_x\varphi(s, \varphi(t, x))R^0(\varphi(t, x)) ds \\ &= - \int_0^\infty \psi(s) \frac{d}{ds} \left[ (\mathcal{X}(K^0 + K^>) - \mathcal{X}^0(K^0) - AK^>) \circ \varphi(s + t, x) \right] ds. \end{aligned}$$

We assume that  $K^>(x) = O(|x|^{L+1})$  and hence  $(\mathcal{X}(K^0 + K^>) - \mathcal{X}^0(K^0) - AK^>)(x) = O(|x|^{L+1})$ . Hence, by (10.12) all integrals above are well defined and there is no boundary term at  $\infty$  when we integrate by parts. Integrating by parts we obtain

$$\begin{aligned} DK^>(\varphi(t, x))R^0(\varphi(t, x)) & \tag{10.28} \\ &= \left( \mathcal{X}(K^0 + K^>) - \mathcal{X}^0(K^0) - AK^> \right) \circ \varphi(t, x) \\ &\quad - A \int_0^\infty \psi(s) \left( \mathcal{X}(K^0 + K^>) - \mathcal{X}^0(K^0) - AK^> \right) \circ \varphi(s + t, x) ds \\ &= \left( \mathcal{X}(K^0 + K^>) - \mathcal{X}^0(K^0) \right) \circ \varphi(t, x) \\ &= \left( \mathcal{X}(K^0 + K^>) - DK^0 \cdot R^0 \right) \circ \varphi(t, x). \end{aligned}$$

Evaluating (10.28) at  $t = 0$  we get that  $K = K^0 + K^>$  satisfies  $\mathcal{X} \circ K = DK \cdot R^0$ , that is, (10.21).

For  $r > L$  we introduce the space

$$C_L^r(\bar{B}_\rho(0)) = \{f \in C^r(\bar{B}_\rho(0)) \mid D^j f(0) = 0, 0 \leq j \leq L\},$$

with the norm  $\|f\| := \max_{L+1 \leq j \leq r} \sup_{x \in \bar{B}_\rho(0)} \|D^j f(x)\|$ . Note that if  $f \in C_L^r(\bar{B}_\rho(0))$  we have

$$\|D^j f(x)\| \leq c_j \|f\| |x|^{L+1-j}, \quad 0 \leq j \leq L+1, \tag{10.29}$$

with  $c_j = 1/(L+1-j)!$ .

We define  $\mathcal{V}$  depending on  $N^{\leq}$ ,  $N^>$  and  $K^>$  by  $\mathcal{V}(N^{\leq}, N^>, K^>)$  to be the left-hand side of (10.27):

$$\begin{aligned} \mathcal{V}(N^{\leq}, N^>, K^>) & \tag{10.30} \\ &= K^>(x) + \int_0^\infty \psi(t) \left[ \mathcal{X}(K^0 + K^>) - \mathcal{X}^0(K^0) - AK^> \right] (\varphi(t, x)) dt, \end{aligned}$$



where  $\mathcal{X} = A + N^{\leq} + N^{>}$  and  $\mathcal{X}^0 = A + N^{\leq}$ . Let  $\Sigma$  be the space of polynomials defined at the beginning of Section 9.4.

**Proposition 10.8.** *Under the previous conditions, we have:*

- (a) *The operator  $\mathcal{V}: V \subset \Sigma \times C_L^{r+1}(\bar{B}_2(0)) \times C_L^r(\bar{B}_1(0)) \rightarrow C_L^r(\bar{B}_1(0))$  defined by (10.30) is continuous with respect to all the three variables and  $C^1$  with respect to  $K^{>}$  in a neighborhood of  $V$  of  $(0, 0, 0)$ .*
- (b) *We have  $\mathcal{V}(0, 0, 0) = 0$  and*

$$D_3\mathcal{V}(0, 0, 0)\Delta = \Delta.$$

**Proof.** The operator  $\mathcal{V}$  is the identity plus  $\mathcal{W}$ . The latter can be written as the composition  $\mathcal{W}_1(\mathcal{W}_2(N^{\leq}), \mathcal{W}_3(N^{\leq}, N^{>}, K^{>}))$ , where

$$\mathcal{W}_3(N^{\leq}, N^{>}, K^{>}) = (A + N^{\leq} + N^{>}) \circ (K^0 + K^{>}) - (A + N^{\leq}) \circ K^0 - AK^{>},$$

$\mathcal{W}_2$  sends  $N^{\leq}$  to the flow  $\varphi(t, x)$  of  $x' = A_E x + N^{\leq}(x)$ , and

$$\mathcal{W}_1(\varphi, g) = \int_0^\infty \psi(s)g(\varphi(s, x))ds. \quad (10.31)$$

We recall that, as in the previous section,  $\Sigma \rightarrow H_3^1$  sending  $N^{\leq}$  to  $K^0$  is analytic,  $H_3^1 \rightarrow C_0^r(\bar{B}_1(0))$  sending  $K^0$  to itself is  $C^\infty$ , and  $\Sigma \rightarrow C_0^{r+1}(\bar{B}_2(0))$  sending  $N^{\leq}$  to itself is  $C^\infty$ . By the results of [dlLO99] and the argument in Proposition 9.8 the operator

$$\tilde{\mathcal{W}}_3: V_3 \subset C_1^{r+1}(\bar{B}_2(0)) \times C_L^{r+1}(\bar{B}_2(0)) \times C_0^r(\bar{B}_1(0)) \times C_L^r(\bar{B}_1(0)) \rightarrow C^r(\bar{B}_1(0))$$

defined by  $\tilde{\mathcal{W}}_3(N^{\leq}, N^{>}, K^0, K^{>}) = (A + N^{\leq} + N^{>}) \circ (K^0 + K^{>}) - (A + N^{\leq}) \circ K^0 - AK^{>}$  is of class  $C^1$ . Then  $\mathcal{W}_3$  is  $C^1$ , takes values in  $C_L^r(\bar{B}_1(0))$  and

$$D_3\mathcal{W}_3(N^{\leq}, N^{>}, K^{>})\Delta = D\mathcal{X}(K^0 + K^{>})\Delta - A\Delta.$$

In particular,  $D_3\mathcal{W}_3(0, 0, 0) = 0$ .

The proposition will be proved once we establish the regularity of  $\mathcal{W}_1$  and  $\mathcal{W}_2$ , that we do in the next lemmas.  $\square$

To study the regularity of  $\mathcal{W}_1$  and  $\mathcal{W}_2$  we introduce the space

$$\Gamma^r = \left\{ \varphi: [0, \infty) \times \bar{B}_1(0) \rightarrow E \mid \varphi \in C^0, \varphi(t, \cdot) \in C^r, \right. \\ \left. \max_{0 \leq j \leq r} \sup_{t, x} e^{-(\mu_+ + \varepsilon)t} |D_x^j \varphi(t, x)| < \infty \right\}$$

with the norm  $\|\varphi\|_\Gamma := \max_{0 \leq j \leq r} \sup_{t,x} e^{-(\mu_+ + \varepsilon)t} \|D_x^j \varphi(t, x)\|$ . It is a Banach space.

**Lemma 10.9.** *The map  $\mathcal{W}_2: V_2 \subset \Sigma \rightarrow \Gamma^r$  which sends  $R^N$  to  $\varphi$ , where  $\varphi(t, x)$  is the solution of  $x' = A_E x + R^N(x)$  is well defined in a neighborhood  $V_2$  of 0 and it is continuous.*

From the basic theory of ordinary differential equations we know that  $\varphi(t, x)$  depends continuously on  $R^N$ , in the sense that if  $a$  is the vector of the coefficients of  $R^N$ ,  $\varphi(t, x, a)$  is continuous. However the lemma states that the continuity holds with respect to the norm in  $\Gamma^r$ . We will use the following version of Gronwall's lemma: if  $u: [0, b) \rightarrow \mathbb{R}$  is continuous  $v: [0, b) \rightarrow \mathbb{R}$  is differentiable, with  $v(0) = 0$ ,  $\alpha \geq 0$  and  $u(t) \leq v(t) + \alpha \int_0^t u(s) ds$  for  $t \in [0, b)$  then

$$u(t) \leq \int_0^t v'(s) e^{\alpha(t-s)} ds, \quad t \in [0, b).$$

**Proof of Lemma 10.9.** First we note that the map  $\Sigma \rightarrow \Sigma$  which sends  $N^\leq$  to  $R^N$  is analytic. From Lemma 10.7 we know that if  $\|R^N\|_{C^{r+1}} \leq \varepsilon/2$  then  $|\varphi(t, x)| \leq e^{(\mu_+ + \varepsilon)t} |x|$ ,  $\|D_x \varphi(t, x)\| \leq e^{(\mu_+ + \varepsilon)t}$  and, for  $j \geq 2$ ,  $\|D_x^j \varphi(t, x)\| \leq M_j e^{(\mu_+ + \varepsilon)t}$ , for all  $x \in \bar{B}_1(0)$  and  $t \geq 0$ .

Let  $R, \tilde{R} \in \bar{B}_{\varepsilon/2}(0) \subset C_1^{r+1}(\bar{B}_1(0))$  and let  $\varphi, \tilde{\varphi}$  be the associated flows. Assume that  $\|R - \tilde{R}\|_{C^{r+1}} \leq \delta$ . We write

$$\varphi(t, x) = e^{A_E t} x + \int_0^t e^{A_E(t-s)} R(\varphi(s, x)) ds$$

and the analogous formula for  $\tilde{\varphi}$ . Subtracting both equations we have

$$\tilde{\varphi}(t, x) - \varphi(t, x) = \int_0^t e^{A_E(t-s)} [\tilde{R}(\tilde{\varphi}) - R(\tilde{\varphi}) + R(\tilde{\varphi}) - R(\varphi)] ds.$$

Multiplying both sides by  $e^{-(\mu_+ + \varepsilon/2)t}$ , using that  $|\tilde{R}(\tilde{\varphi}(s, x)) - R(\tilde{\varphi}(s, x))| \leq c \|\tilde{R} - R\|_{C^2} |\varphi(s, x)|^2$  and taking norms we get

$$\begin{aligned} & e^{-(\mu_+ + \varepsilon/2)t} |\tilde{\varphi}(t, x) - \varphi(t, x)| \\ & \leq \int_0^t c \delta e^{(\mu_+ + (3/2)\varepsilon)s} ds + (\varepsilon/2) \int_0^t e^{-(\mu_+ + \varepsilon/2)s} |\tilde{\varphi}(s, x) - \varphi(s, x)| ds \end{aligned}$$

and by Gronwall's lemma,  $e^{-(\mu_+ + \varepsilon/2)t} |\tilde{\varphi}(t, x) - \varphi(t, x)| \leq c\delta e^{(\varepsilon/2)t} (-1/(\mu_+ + \varepsilon))$ . Hence

$$|\tilde{\varphi}(t, x) - \varphi(t, x)| \leq \frac{c\delta}{-(\mu_+ + \varepsilon)} e^{(\mu_+ + \varepsilon)t}.$$

Proceeding in the same way from

$$D_x \varphi(t, x) = e^{A_E t} + \int_0^t e^{A_E(t-s)} D R(\varphi(s, x)) D_x \varphi(s, x) ds$$

and the analogous formula for  $D_x \tilde{\varphi}(t, x)$  we arrive at

$$\|D_x \tilde{\varphi}(t, x) - D_x \varphi(t, x)\| \leq \frac{\delta + (\varepsilon/2) \|\tilde{\varphi}(t, x) - \varphi(t, x)\|_\Gamma}{-(\mu_+ + \varepsilon)} e^{(\mu_+ + \varepsilon)t}.$$

If  $k \geq 2$  we proceed inductively starting from

$$D_x^k \varphi(t, x) = \int_0^t e^{A(t-s)} \sum_{j=1}^k \sum_{i's} c D^j R(\varphi(s, x)) D_x^{i_1} \varphi(s, x) \dots D_x^{i_j} \varphi(s, x) ds$$

and using the analogous manipulations as in the cases  $k = 0, 1$  we get

$$\|D_x^k \tilde{\varphi}(t, x) - D_x^k \varphi(t, x)\| \leq \delta c_k e^{(\mu_+ + \varepsilon)t},$$

where  $c_k$  are positive constants independent of  $\varphi$  and  $\tilde{\varphi}$ .  $\square$

**Lemma 10.10.** *The map  $\mathcal{W}_1 : V_1 \subset \Gamma^r \times C_L^r(\bar{B}_1(0)) \rightarrow C_L^r(\bar{B}_1(0))$  defined by (10.31) is well defined in a neighborhood  $V_1$  of  $(0, 0)$ , is continuous with respect to both variables, is  $C^1$  with respect to  $g$ , and*

$$D_2 \mathcal{W}_1(\varphi, g) \Delta = \int_0^\infty \psi(s) \Delta(\varphi(s, x)) ds. \quad (10.32)$$

**Proof.** Throughout the proof  $C$  will mean a constant independent on the functions, which may take different values in different places. We write

$$\begin{aligned} & \mathcal{W}_1(\tilde{\varphi}, \tilde{g}) - \mathcal{W}_1(\varphi, g) \\ &= \int_0^\infty \psi(s) [\tilde{g}(\tilde{\varphi}(s, x)) - g(\tilde{\varphi}(s, x))] ds + \int_0^\infty \psi(s) [g(\tilde{\varphi}(s, x)) - g(\varphi(s, x))] ds. \end{aligned}$$

Let  $L + 1 \leq k \leq r$ . The  $D_x^k$  derivative of  $\psi(s)[\tilde{g}(\tilde{\varphi}(s, x)) - g(\tilde{\varphi}(s, x))]$  is bounded by

$$\begin{aligned} & \left\| \psi(s) \sum_{i=1}^k \sum_{\substack{1 \leq \ell_1, \dots, \ell_i \leq k \\ \ell_1 + \dots + \ell_i = k}} C[D^i \tilde{g}(\tilde{\varphi}(s, x)) - D^i g(\tilde{\varphi}(s, x))] \right. \\ & \quad \left. \times D_x^{\ell_1} \tilde{\varphi}(s, x) \cdots D_x^{\ell_i} \tilde{\varphi}(s, x) \right\| \\ & \leq M e^{(-\lambda_- + \varepsilon)s} \sum_{i=1}^k \sum_{\ell'_s} C \|\tilde{g} - g\|_{C^k} |\tilde{\varphi}(s, x)|^{(L-i+1)_+} \\ & \quad \times \|D_x^{\ell_1} \tilde{\varphi}(s, x)\| \cdots \|D_x^{\ell_i} \tilde{\varphi}(s, x)\| \\ & \leq M e^{(-\lambda_- + \varepsilon)s} \sum_{i=1}^L \sum_{\ell'_s} C \|\tilde{g} - g\|_{C^k} e^{(\mu_+ + \varepsilon)(L-i+1)s} |x|^{(L-i+1)_+} \\ & \quad \times M_{\ell_1} e^{(\mu_+ + \varepsilon)s} \cdots M_{\ell_i} e^{(\mu_+ + \varepsilon)s} \\ & \leq C e^{[-\lambda_- + \varepsilon + (\mu_+ + \varepsilon)(L+1)]s} \|\tilde{g} - g\|_{C^k}, \end{aligned}$$

since  $(L - i + 1)_+ + i \geq L + 1$ .

The  $D_x^k$  derivative of  $\psi(s)[g(\tilde{\varphi}(s, x)) - g(\varphi(s, x))]$  is bounded by

$$\begin{aligned} & \left\| \psi(s) \sum_{i=1}^k \sum_{\substack{1 \leq \ell_1, \dots, \ell_i \leq k \\ \ell_1 + \dots + \ell_i = k}} C[D^i g(\tilde{\varphi}(s, x)) D_x^{\ell_1} \tilde{\varphi}(s, x) \cdots D_x^{\ell_i} \tilde{\varphi}(s, x) \right. \\ & \quad \left. - D^i g(\varphi(s, x)) D_x^{\ell_1} \varphi(s, x) \cdots D_x^{\ell_i} \varphi(s, x)] \right\|. \end{aligned}$$

Now, by adding and subtracting appropriate terms, we get the desired bounds. We have to deal with terms  $[D^i g(\tilde{\varphi}(s, x)) - D^i g(\varphi(s, x))] D_x^{\ell_1} \tilde{\varphi}(s, x) \cdots D_x^{\ell_i} \tilde{\varphi}(s, x)$  which are bounded by

$$\begin{aligned} & \|g\|_{C^{L+1}} e^{(\mu_+ + \varepsilon)(L+1-i-1)s} |\tilde{\varphi}(s, x) - \varphi(s, x)| \|D_x^{\ell_1} \tilde{\varphi}(s, x)\| \cdots \|D_x^{\ell_i} \tilde{\varphi}(s, x)\| \\ & \leq C e^{(\mu_+ + \varepsilon)(L+1)s} \|\tilde{\varphi} - \varphi\|_{\Gamma}, \end{aligned}$$

if  $i < L$ , by

$$\begin{aligned} & \|g\|_{C^{i+1}} |\tilde{\varphi}(s, x) - \varphi(s, x)| \|D_x^{\ell_1} \tilde{\varphi}(s, x)\| \cdots \|D_x^{\ell_i} \tilde{\varphi}(s, x)\| \\ & \leq C \|\tilde{\varphi} - \varphi\|_{\Gamma} e^{(\mu_+ + \varepsilon)s} e^{(\mu_+ + \varepsilon)is}, \end{aligned}$$

if  $L \leq i < r$ , and by  $Ce^{(\mu_+ + \varepsilon)rs} \omega(\|\tilde{\varphi} - \varphi\|_{C^0})$ , where  $\omega$  is the modulus of continuity of  $D^r g$ , by the uniform continuity of  $D^r g$  on  $\tilde{B}_1(0)$ .

We also need to control the terms of the form

$$D^j g(\varphi) D_x^{\ell_1} \tilde{\varphi} \cdots [D_x^{\ell_m} \tilde{\varphi} - D_x^{\ell_m} \varphi] \cdots D_x^{\ell_i} \varphi,$$

which are bounded by

$$\begin{aligned} & \|g\|_{C^{i+1}} |\varphi(s, x)|^{L+1-i-1} C e^{(\mu_+ + \varepsilon)(i-1)s} \|D_x^{\ell_m} \tilde{\varphi} - D_x^{\ell_m} \varphi\|_{\Gamma} e^{(\mu_+ + \varepsilon)s} \\ & \leq C \|g\|_{C^{i+1}} e^{(\mu_+ + \varepsilon)(L+1)s} \|\tilde{\varphi} - \varphi\|_{\Gamma}, \end{aligned}$$

if  $i < L$  and analogous bounds in the other cases, as we have got for the previous terms.

When we integrate from 0 to  $\infty$ , we take the supremum over  $x$  and the maximum over  $k$ , we obtain the continuity in the topologies we are working with.

To prove that  $\mathcal{W}_1$  is differentiable with respect to  $g$  we only have to check that it is a bounded linear operator in  $g$ . This follows immediately taking  $\tilde{g} = 0$  and  $\tilde{\varphi} = 0$  in the previous estimates. In such a way we get

$$\|\mathcal{W}_1(g)\|_{C_L^r} \leq C \|g\|_{C_L^r}.$$

To study the continuity of  $D_2 \mathcal{W}_1(\varphi, g)$ , in view of formula (10.32) we have to do the same kind of estimates as we have done when dealing with  $\psi(s)[\tilde{g}(\tilde{\varphi}(s, x)) - g(\tilde{\varphi}(s, x))]$  but changing  $g$  by  $\Delta \in \Gamma^r$ .  $\square$

The end of the proof of Theorem 3.2 in the differentiable case follows in a completely analogous way as in the end of the proof of Theorem 3.1. Applying the generalized version of the implicit function theorem (see [Nir01]) to  $\mathcal{V}(N^{\leq}, N^>, K^>) = 0$  near  $(0, 0, 0)$  we get a continuous map  $K^> = \mathcal{V}^*(N^{\leq}, N^>)$  defined in a neighborhood of  $(0, 0)$ .

Given a vector field  $\mathcal{X}$  satisfying the hypotheses of Theorem 3.2, we scale it to  $\mathcal{X}^\delta = A + N^{\leq, \delta} + N^{>, \delta}$  with  $\delta$  so small that  $(N^{\leq, \delta}, N^{>, \delta})$  belongs to the domain of  $\mathcal{V}^*$ . The parameterization  $K = K^0 + K^>$  thus obtained is the solution we are looking for.

**Remark 10.11.** Note the remarkable similarities between the proofs of Theorems 3.1 and 3.2. Also the analogy of the argument in Proposition 9.3 which computes the spectrum of the operators  $\mathcal{L}_{A,B}^i$  defined in (9.4) and the one in Proposition 10.2 which computes the spectrum of the operators  $\tilde{\mathcal{L}}_{A,B}^i$  defined in (10.5). Similarly, their use of the recursive solution to the hierarchy of equations for the low order terms is completely analogous.

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## Appendix A. Remarks on cohomology equations and non-uniqueness of invariant manifolds

Since cohomology equations play an important role in this theory, it is interesting to give a heuristic guide to their solution.

A cohomology equation is an equation for  $\Delta$  of the form

$$M(x)\Delta(x) - \Delta \circ R(x) = \eta(x), \quad (\text{A.1})$$

where  $\Delta$  and  $\eta$  are vector-valued functions,  $M$  is a function taking values on the space of linear operators and  $R$  is a diffeomorphism. When facing an equation of the form (A.1), it is natural to try to isolate  $\Delta$  explicitly from one term and iterate the resulting expression.

If we isolate  $\Delta$  from the second term in (A.1), we are left with

$$\Delta(x) = -\eta \circ R^{-1}(x) + M \circ R^{-1}(x) \Delta \circ R^{-1}(x), \quad (\text{A.2})$$

which, upon iteration, leads to

$$\begin{aligned} \Delta = & -\eta \circ R^{-1} - M \circ R^{-1} \eta \circ R^{-2} - M \circ R^{-1} M \circ R^{-2} \eta \circ R^{-3} \\ & - \dots - [M \circ R^{-1} M \circ R^{-2} \dots M \circ R^{-n}] \eta \circ R^{-n-1} \\ & + [M \circ R^{-1} M \circ R^{-2} \dots M \circ R^{-n-1}] \Delta \circ R^{-n-1}. \end{aligned} \quad (\text{A.3})$$

If we isolate  $\Delta$  from the first term in (A.1), we are left with

$$\Delta(x) = M^{-1}(x)\eta(x) + M^{-1}(x)\Delta \circ R(x), \quad (\text{A.4})$$

which, upon iteration, leads to

$$\begin{aligned} \Delta = & M^{-1}\eta + M^{-1}M^{-1} \circ R \eta \circ R \\ & + \dots + [M^{-1}M^{-1} \circ R \dots M^{-1} \circ R^n] \eta \circ R^n \\ & + [M^{-1}M^{-1} \circ R \dots M^{-1} \circ R^n] \Delta \circ R^{n+1}. \end{aligned} \quad (\text{A.5})$$

Note that the general term in both (A.3) and (A.5) consists of the multiplication by a large number of linear operators applied to  $\eta$  composed by the right with a high iterated of  $R$  or of  $R^{-1}$ . These sums can be shown to converge as  $n \rightarrow \infty$  in two different cases by two different arguments.

In the first argument we use that if  $M$  is a contraction, then (A.3) will converge in the  $\|\cdot\|_{C^0}$  norm. In the problems considered in this paper the point is that if  $M = DF \circ K$  is a contraction then  $R$ , which agrees at first order with it close to the fixed point, will also be a contraction. Hence,  $R^{-1}$  is expansive. Since (A.3) involves composing with  $R^{-1}$ , this will require that the functions are defined everywhere. If it is not the case it requires performing extensions, etc. We note that performing extensions of  $R$  causes that the resulting manifold may depend on the extension procedure, so that the local results will be quite non-unique. This study is the basis of the results in Section 6.

The second argument uses that the  $R^n(x)$  converges to a point as  $n \rightarrow +\infty$ . This happens when  $\|DR\|_{C^0} < 1$  in a neighborhood of the origin. In this case,  $M^{-1}$  will be an expansion. To have convergence of the right-hand side of (A.5) we will need that the operator  $\Delta \mapsto \Delta \circ R$  is a strong enough contraction to overcome the expansion caused by  $M^{-1}$ . The basic idea to obtain contraction that gives sense to (A.5) is to work in a space of functions  $\Delta$  defined on  $U$  such that  $\sup_{x \in U} |x|^{-L} |\Delta(x)| < \infty$  and to use the weighted norm

$$\|\Delta\| := \sup_{x \in U} |x|^{-L} |\Delta(x)|.$$

With this norm, we have

$$\begin{aligned} \|\Delta \circ R\| &= \sup_{x \in U} |x|^{-L} |R(x)|^L |R(x)|^{-L} |\Delta \circ R(x)| \\ &\leq \sup_{x \in U} |x|^{-L} |R(x)|^L \cdot \sup_{x \in U} |R(x)|^{-L} |\Delta \circ R(x)| \\ &\leq \|DR\|_{C^0}^L \|\Delta\|. \end{aligned} \tag{A.6}$$

For low  $L$ , to work in such spaces is quite natural and we easily have the contractive property. This is the basis of the results in Section 8.

For high  $L$ , we have to resort to other methods. We observe that, under appropriate conditions, one can obtain the low-order terms of the problem matching derivatives and then, obtain the remainder using this method. This is the basis of the results in Section 9. Note that solving the problem for the low-order terms requires non-resonance conditions.

We note that in both cases, we obtain uniqueness of the solution in the corresponding space (after having fixed an extension of  $R^{-1}$  in the first case).

Nevertheless, it is quite important to note that, even in the case that both solutions (A.3), (A.5) make sense, they may fail to be the same. Note that, even if one had equal solutions for a certain  $\eta$ , adding a small bump to  $\eta$  causes perturbations that go

towards the origin in (A.3), and that go towards infinity in (A.5) and hence, for this perturbed  $\eta$  the solutions given by (A.3) and (A.5) will be different.

One important difference between the methods of solution is that, if we take derivatives of the general term in (A.3), we pick factors  $DR^{-n}$ , which are growing. However in (A.5), we obtain factors  $DR^n$  which are decreasing. Hence, if the series of the  $k$ -derivatives of the terms in (A.5) converge, the series of the  $j$ -derivatives also converge for values of  $j$  from  $k$  to the degree of differentiability of  $\eta$ . The solutions produced by (A.5), as soon as they start converging, they have all the derivatives that  $\eta$  has. On the other hand, those produced by (A.3) only have a finite number of derivatives that cannot be improved by assuming more differentiability of  $\eta$ .

Unfortunately, the regularity that can be produced automatically by (A.3) is always smaller than that allowed by bootstrap using (A.5).

All the above phenomena have a correspondence in the theory of invariant manifolds. The fact that slow manifolds with low regularity are not unique has been in the literature for a certain time. If one makes hypotheses that imply that there is certain growth at infinity, one can readily show uniqueness. This was also known using Irwin's method [dILW95]. On the other hand, under non-resonance assumptions, one can get uniqueness under moderate differentiability assumptions.

That is, among all the rough invariant manifolds tangent to the space, there is one which is moderately differentiable, and this moderately differentiable manifold is as smooth as the map.

Examples that show that the manifolds with good behavior at infinity do not agree with the moderately smooth ones have been constructed in [dIL97].

## Appendix B. Historical remarks and information on the literature on non-resonant invariant manifolds

In this section we have collected some references on the problem of invariant manifolds associated to subspaces in the stable part of the spectrum.

In contrast to the very vast literature on stable manifolds—for which a similar attempt would be beyond the capacity of the authors—the literature on invariant manifolds associated to smaller sets of the spectrum is much more limited. Of course, this attempt cannot be considered a definitive effort (for instance, we have not been able to trace the work of Darboux, which is mentioned by Poincaré and Lyapunov). We can only hope that our modest search can inspire others to do a more thorough job.

### B.1. Early history

It seems to us that one-dimensional invariant submanifolds were more or less known in the analytic case, and with resonance conditions somewhat stronger than those considered in the present paper.

It seems well accepted that some versions of invariant manifold theory, at least for the analytic case, were known to Darboux, Poincaré and Lyapunov. Unfortunately, we



have not been able to locate the works of Darboux, but we will comment on some works of Poincaré and Lyapunov.

### B.2. Two results of Poincaré

One of us (R.L.) learned about the existence of [Poi90] (reproduced in [Poi50]) from conversations with D. Ruelle in the early 1980s.

The motivation for [Poi90] was the theory of special functions.

When  $F$  is a polynomial and  $E = \mathbb{C}$ , the equation

$$F \circ K(t) = K(\lambda t) \quad (\text{B.1})$$

can be interpreted as saying that the system of functions given by the components of  $K$  admits a multiplication rule (*théoreme de multiplication*). Examples of such systems of functions (or systems satisfying the closely related addition rules) are the trigonometric functions and the elliptic functions. For instance,  $K(\theta) = (\sin \theta, \cos \theta)$  satisfies  $K(2\theta) = F(K(\theta))$ , where  $F(x, y) = (2xy, y^2 - x^2)$ . Note that  $F(0, 1) = (0, 1)$ . Similar formulas for the duplication of the argument are known for elliptic integrals. The fact that there are duplication formulas is related to the solvability of the quintic using elliptic functions and their inverses.

The paper [Poi90] shows that, given a map  $F$  and provided that  $\lambda$ ,  $|\lambda| > 1$ , is a simple eigenvalue of  $DF(0)$  and that there are no eigenvalues of  $DF(0)$  which are powers of  $\lambda$ , one can find a formal series for  $K$ . Moreover, using the majorant method, one can show that the formal series for  $K$  converges.

The paper also contains the interesting observation (see [Poi50, p. 541]) that when  $F$  is a polynomial, every function  $K$  satisfying (B.1) is entire. The reason is that, when  $F$  is a polynomial, the functional Eq. (B.1) forces the domain of definition of  $K$  to be invariant under multiplication by  $\lambda$ . Hence, if it contains a ball, it is the whole complex plane. We note that this observation generalizes without difficulty to the situation when  $F$  is an entire function and we are working on a Banach space.

In [Poi90], Poincaré also studies the case when  $F^{-1}$  is a rational transformation, that he calls *Cremona*. In this case, he makes some dynamical observations. For instance, in [Poi50, bottom half of p. 561], he relates the question of existence of solution to whether the iterates of the transformation converge to a fixed point—this is indeed the dynamical characterization of invariant manifold.

From a more dynamical point of view, similar series were considered in [Poi87], where all Chapter VII is devoted to asymptotic expansions around periodic solutions of periodic vector fields. Taking time- $T$  maps, this problem reduces to the setting about maps that we have considered in this paper. The logarithms of the eigenvalues of the time- $T$  map are called *exposants caractéristiques*. In modern language, they are the Floquet exponents. Note that what we would call today Lyapunov exponents (which can be considered in more general settings than periodic systems) are, in the case of periodic systems, the real part of Poincaré's *exposants caractéristiques*. More confusingly, in the translation of Lyapunov that we have used, the name *characteristic exponent* refers

to the negative of what we call now Lyapunov exponent. This would be, of course, the negative of the real part of the *exposant caractéristique* for the particular case of periodic systems.

In [Poi87] the crucial paragraphs dealing with stable and unstable manifolds are 104 and 105. In paragraph 104, under the assumption that there are no resonances (the non-resonance condition is the last formula of paragraph 104), it is shown that one can obtain a formal power series expansion of exponentials with arbitrary constants.

The convergence of the series is studied in 105. The first paragraph asserts the convergence of the series of expansions in powers of the exponential under the assumption that the eigenvalues belong to what we now call the Poincaré domain (i.e., when the convex hull of the eigenvalues does not include zero). Of course, the reason why this condition enters is that, for eigenvalues satisfying these conditions, the small divisors that appear are bounded away from zero.

We note that, even if it is not said explicitly, the condition that the eigenvalues are different is indeed assumed. The proof of convergence is rather succinct. Nevertheless, it should have been quite clear to Poincaré and his contemporaries since it is very similar to arguments that had been done in detail in his thesis [Poi79] (reproduced in [Poi16]).

From the point of view of invariant manifold theory, the last paragraph of page 339 is quite interesting. Here, Poincaré discusses the case when there are stable and unstable characteristic exponents at the same time. He observes that the series for  $K$  remains convergent if one sets to zero the constants corresponding to coordinates along the expanding or neutral eigendirections. The arguments here are somewhat skimpy, but a modern mathematician can supply the missing details without too much trouble. One is left with a set of solutions which tend to zero parameterized by as many constants as stable directions. This is, of course, our modern stable manifold. A similar construction works for the unstable solutions. Poincaré called these solutions *solutions asymptotiques*.

The rest of Chapter VII contains a variety of expansions of these sets of solutions. It includes, quite notably, the expansions in terms of a slow parameter, which are then shown to be divergent. Of course, much modern work is still being done in these slow perturbations and related areas.

### B.3. The work of Lyapunov

In Chapters 11–33 [Lya92] (see also the summary in Chapter 3 and the proofs of convergence in Chapter 23), Lyapunov introduces the method of arbitrary constants, which consists in finding exponential solutions with arbitrary constants. Since the constants do not evolve in time, this is closely related to the problem of linearization (compare the expansions of the system studied and those of the linear systems). Invariant manifolds can be obtained by setting some of the constants to zero.

One important difference between [Lya92] and [Poi90,Poi87] is that [Lya92] considers systems which are *regular* (roughly, the definition is that the forward and backward Lyapunov exponents agree). This is a more general setting than that of periodic systems. In the case of regular systems, [Lya92] contains expansions of the solutions in

terms of arbitrary constants. The derivation of the formal expansions in [Lya92] does not need non-resonance conditions.

In Chapter 23 of [Lya92], the question of convergence of these formal expansions is studied. This is done under the condition that there are no resonances and no repeated eigenvalues, and that all the eigenvalues are stable or unstable. Here one can find a note giving credit to [Poi79] for dealing with the more general case of the Poincaré domain.

In particular, we call attention to Theorem II of Section 24, which is a complete statement of the strong stable manifold theorem for analytic systems (see also Theorem II of Chapter 13).

One interesting remark of Lyapunov in Chapter 11 is that one can consider families that correspond to any subset of eigenvalues. This amounts to setting to zero a subset of the arbitrary constants used in the expansion. This is hard to interpret from the dynamical point of view since the arbitrary constants do not have a dynamical interpretation. In particular, the set obtained setting them to zero does not need to be invariant. Of course, setting to zero all the non-decreasing modes is an invariant set, as pointed out by Poincaré. With modern insight, setting to zero all the modes that are non-decreasing or decreasing more slowly than a certain rate is invariant. Indeed, it is the strongly stable manifold. As it was shown in examples in [dIL97], in general one cannot get invariant manifolds tangent to a subspace if there are resonances of the type we have excluded in the present paper.

Overall, one cannot be but surprised by the enormous similarities in the problems and in the results between the contents of these chapters in [Lya92] and the corresponding ones of the book by Poincaré, which appeared in the same year. Of course, there are big differences in style and in the methods as well as in the way that proofs are presented.

A modern exposition of some of the convergence results of Lyapunov can be found in [Lef77, V.4]. It contains a statement and a proof of the expansion in arbitrary constants under non-resonance assumptions and provided that all eigenvalues are stable, and that the linearization is a constant (we remark that using Floquet theory, one can reduce the periodic case to the constant case). We have not been able to locate in any of these classical works the consideration of resonant terms.

#### *B.4. Modern work*

It seems that the particular case of one-dimensional stable invariant manifolds (when there are no resonances) has appeared several times in the modern literature.

The papers [FR81] and [FG92] use the parameterization method for one-dimensional manifolds, specially in conjunction with numerical analysis. They establish not only convergence of the series involved, but they also estimate the errors incurred when using a numerical approximation. Indeed, both papers have taken care of estimating actually the roundoff error so that a finite calculation can establish facts about transversality of intersections, etc.

It seems to us that similar results could be obtained using the functional equations (2.1) and the theory developed in the present article.

Numerical work for higher dimensional maps has been studied in [BK98], which undertook the task of systematically computing Taylor expansions of invariant manifolds. This could be considered one implementation of our result in Lemma 9.2 for finite-dimensional systems. The authors of [BK98] indeed made the observation that the calculations can be carried out to any order provided that there are no resonances but they leave open the issue of whether these formal calculations are the jet of an invariant object.

We note that our formalism could be used to provide an a posteriori estimate of the error of these numerical calculations. Once a polynomial satisfies (2.1) quite accurately, then it is close to being a fixed point of a map  $\mathcal{N}$  (which is a solution of  $\mathcal{T} = 0$ ). Since  $\mathcal{N}$  is a contraction, there is a fixed point at a distance that can be estimated by the error of the numerical approximation. This is the usual a posteriori estimates of numerical analysis.

The work [Pös86] considers invariant manifolds associated to non-resonant eigenspaces. It also studies the equation of semi-conjugacy of the motion on the manifold to a linear motion. In contrast to the classical works, it can deal with situations when the eigenvalues are not stable and all of the same sign. Small divisors appear, but they can be overcome using the majorant method and improvements of estimates in [Sie42]. It is interesting to note that the non-resonance conditions obtained in [Pös86] are more general than those required by the straightforward application of usual KAM method. For more modern developments, see [Sto94].

The paper [CF94] took up the task of making sense of the one-dimensional manifolds of Poincaré and Lyapunov, in the case of finite differentiability. It reduced the problem to a fixed point equation that was solved by the contraction method. In this paper there was some consideration given to the resonant case, and the result proved was that Eq. (2.1) could be solved if and only if our equations for  $K$  and  $R$  provide a formal solution.

We note that all papers mentioned above seem to require the dynamics in the invariant manifold considered to be linearizable.

The work whose results are most closely related to those of the present paper is [dIL97]. It contains a study of invariant manifolds which does not require the motion on the invariant manifold to be linearizable. In [CFdIL03a, CFdIL03b] and the present paper, we improve the results of [dIL97] by not requiring the splitting to be invariant. Hence, we can associate invariant manifolds to eigenspaces in a non-trivial Jordan form. Also, we can improve the regularity obtained from  $C^{r-1+\text{Lip}}$  to  $C^r$  (this later improvement in the regularity was obtained also in [EIB98]). The interest of this sharp regularity is that the same method that we use to optimize the regularity also allows us to obtain rather sharp results on the dependence on parameters. The methods of [CFdIL03a, CFdIL03b] and those of [dIL97] are quite different since [dIL97] is based on normal forms and on a graph transform method.

There are some generalizations of the non-resonant manifolds considered in [CFdIL03a, CFdIL03b] and in the present paper to some rather general semigroups—which include many parabolic equations—in [dILW97]. The main difference is that in [dILW97], the assumption on invertibility of the linearized operator used in the present paper is dropped, since the linearization of the evolution of an elliptic

equation is a compact operator, hence, not invertible. The paper [FdLM03] considers non-autonomous versions of non-resonant manifolds to orbits whose Lyapunov exponents are non-resonant. Indeed, the conditions of hyperbolicity in [FdLM03] generalize the customary conditions in hyperbolicity—in which one allows some spread on the rate of expansion but requires uniformity—and those in non-uniform hyperbolic systems—in which one requires a single rate, but allows some non-uniformity along the orbit.

### B.5. Slow manifolds in applications

In many applications, one is interested in the phenomena that happen at a slow time scale. The fast phenomena disappear quickly and are not observable. Very often, these slow phenomena are governed by geometric objects that are described as *slow manifolds*. Two sciences in which such phenomena happen and have been considered are atmospheric sciences and chemical kinetics.

It should be kept in mind that the name slow manifolds is overused and that there are many mathematical definitions of slow manifolds which capture the idea of slowness. See, for example [Boy95,Lor92] for a—non-exhaustive—discussion of possible meanings of slow manifolds.

One of the meanings of slow manifolds that have been discussed is precisely the situation described in this paper, namely manifolds in a neighborhood of a fixed point corresponding to slowest eigenvalues (or to a neighborhood of an invariant manifold corresponding to invariant bundles).

Notably, in atmospheric sciences, slow manifolds in a sense very similar to the one considered here have been studied in [Lor86,Jab91].

In chemical kinetics, we mention the method of intrinsic low-dimension manifolds (ILDM) of Maas and Pope [MP92] and the iterative method of Fraser and Roussel [Fra88,RF90]. If the system has an attracting fixed point the method of Fraser–Roussel approximates the slow manifold considered in this paper as a particular case of the non-resonant manifold. However the ILDM is different, among other things because it is not invariant by the flow.

Some more recent references in the chemical literature are [Fra98,Smo91,PR93,WMD96,GKZD00].

It should be emphasized that one should not expect that other sensible definitions of slow manifolds yield the same mathematical objects. For example, if we impose slow growth at infinity, we obtain invariant manifolds [dILW95], which, as shown in [dIL97] are not the same as the non-resonant manifolds considered here. This explains (see [Lor92]) the discrepancy in the title of [Lor86,Lor87].

We call attention to [Jab91], which performs a formal analysis similar to the one introduced here for the models in [Lor87] and invokes a Hartman–Grobman theorem to justify the existence of the objects. Also [KK02] uses asymptotic analysis to understand the difference between the methods commonly used in chemical kinetics.

We think that the idea of non-resonant manifolds is very closely related to ideas that have been proposed in renormalization groups. In [LMS95], it is argued that beta function calculations of renormalization groups correspond to computation of the jets

of smooth invariant manifolds as the ones we consider in Lemma 9.2. Nevertheless, it is also argued that the ones which should be considered in several problems of phase transitions are other slow manifolds.

This is a very nice idea, and it would be quite interesting to make it precise. Of course, the mathematically precise definition of renormalization group as a well defined differentiable operator in a Banach space, has only been achieved in a few cases (mainly dynamical systems and hierarchical models). Even in these cases where the renormalization operator is a bona fide differentiable operator in Banach spaces, its linearization is a compact operator, hence not invertible, and the theorems of this paper do not apply. One has to use those of [dILW97].

In [dIL97] it is shown that the non-coincidence of these manifolds can be used to prove that the invariant circles predicted by the Hopf bifurcation for maps are often not  $C^\infty$ .

For some specialized semi-groups—including the important Navier-Stokes equations—some slow invariant manifolds have been constructed in the remarkable articles [FS84b,FS84a,FS87] (see also the surveys [FS86a,FS86b]). In these works, slow manifolds are constructed under non-resonance conditions. We note in particular that [FS86b] establishes that these invariant manifolds are analytic. This is in contrast with examples in [dILW97] which show that, for semi-groups, the slow manifolds may not be analytic. We thank E. Titi for bringing these papers to our attention.

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